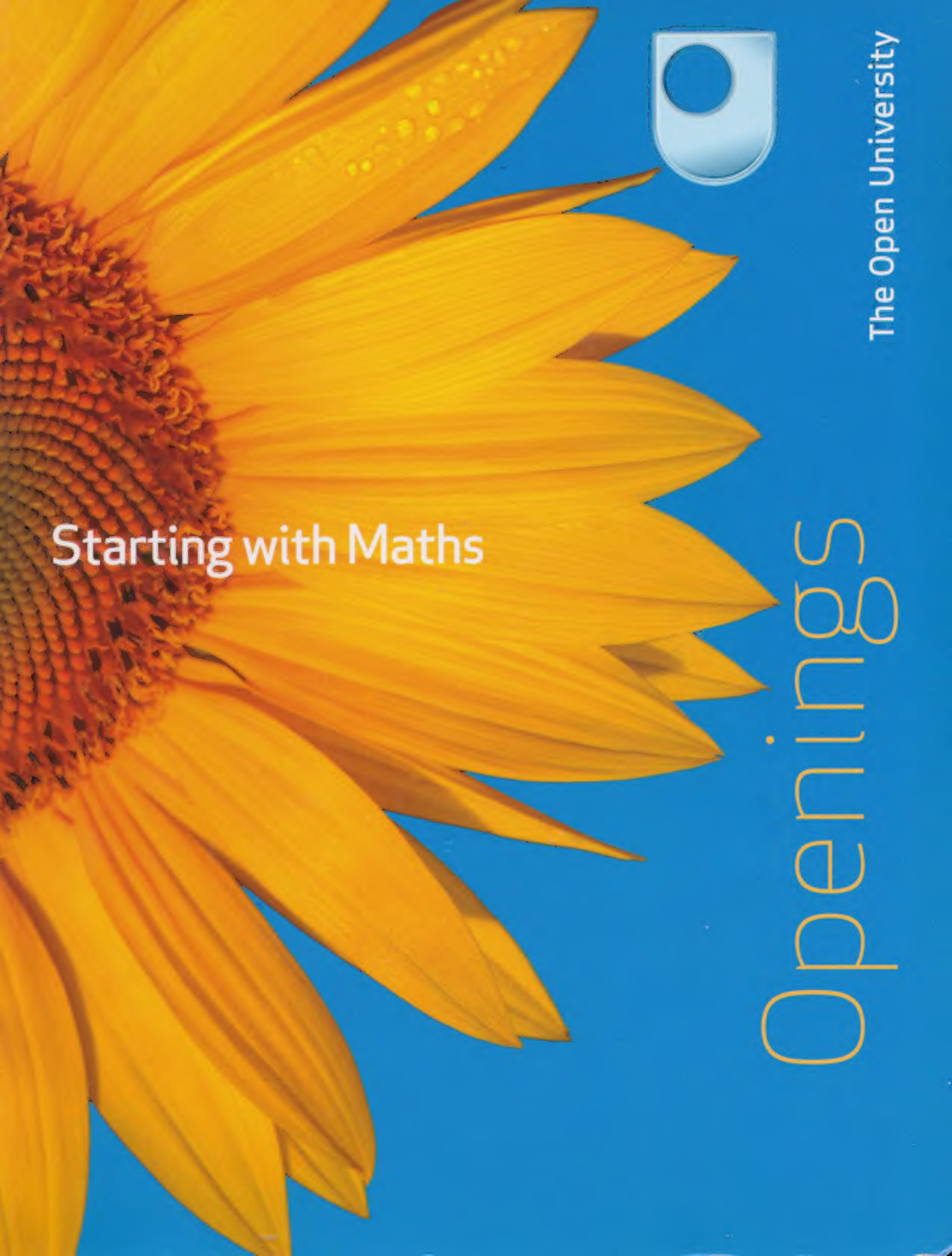

Starting with Maths

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1 Introducing *Starting with Maths*

If you mention to some people that you are studying a maths course, you may well receive a surprised and puzzled look! To many people, mathematics is a subject full of confusing rules and calculations that they think has little relevance to their lives. But just pause a minute – is this really true? Have you used any mathematical skills today or done something that depended on mathematics? You may not even have realised you were doing so. If you have used a train timetable, read an opinion poll, taken some medicine, watched a special effects film or played a computer game, mathematics will have played a part. It underpins almost everything we do. Not only that, but many people do use a lot of mathematical skills already, whether they are keeping tabs on their money or planning the quickest way to get somewhere or tackling a DIY project.

The main aim of this course is to help you feel more confident in using maths in a variety of different situations – at home, at work or in your other studies. Three main themes are developed in the course:

- improving your mathematical skills including using a calculator effectively
- developing problem-solving strategies so that you know what to do when you get stuck
- practising general study skills to help you become an effective learner.

This chapter discusses ways of tackling the course and how people feel about and use mathematics in their lives. Working through this chapter carefully will help prepare you for the detailed mathematics and calculator work in the rest of the course, starting with Chapter 2. It also introduces you to some ways of studying, such as learning from activities, taking notes and using audio tracks. We hope that learning actively in this way will help you both on this course and in the future. You will also see some of the different ways people try to solve problems.

1.1 Developing your mathematical skills

This section describes how to tackle different kinds of activities. The first step is to get organised – find somewhere comfortable to study where you can set out your materials to work on, and then gather together the extra things you need – pens, pencils, eraser, paper, calculator, media player, and so on. To illustrate what we mean by learning actively, try Activity 1.

Activity 1 Do you do Sudoku?

These puzzles are closely linked to an advanced branch of mathematics known as Graph Theory.

In Spring 2005, a puzzle craze known as Sudoku swept across the UK in national newspapers. It involved putting numbers on a square grid and the authors claimed that it needed no mathematics whatsoever, just a logical mind. An example is shown below.

The large square is composed of four blocks, each of which has four smaller squares within. There are also four rows across and four columns down, so it is known as a four by four (4×4) puzzle.

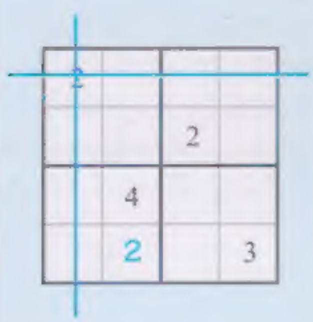
The idea is to arrange the numbers 1, 2, 3 and 4 in each block, so that each row and each column contains only one of the numbers 1, 2, 3 and 4. For example, the bottom left-hand block already has a 4 in it, so you'll need to put 1, 2 and 3 in the remaining cells in that block. Make sure that you do not end up with two numbers the same in a row or in a column though. Have a go! You might find it easiest to write your ideas directly on the diagram here, though you can copy it onto paper if you like. When you have either solved the puzzle or spent about 10 minutes on it, read through the comments below.

Comment

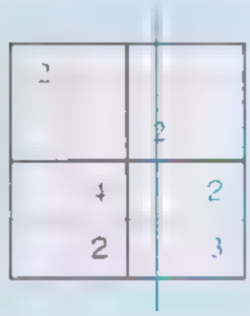
If you have never seen these puzzles before, you might have found this one quite tricky. There are many different ways that you can tackle this problem. The first step is to try to sort out exactly what you are being asked to do and to make sure you understand the problem. You may find it helpful to use a highlighter pen, as we have done here, to emphasise the key bits of information. Then you might like to get a feel for the problem by putting in a few numbers just by guesswork. Unless you have been lucky, you will probably realise fairly soon that this does not work very well, but it will have given you a better idea of what is involved. Starting to get a feel for what might be involved is a very important first stage for any problem.

The next tactic to try is to make the problem a bit simpler by breaking it down into steps, perhaps by concentrating on just one block or on just one kind of number, a 2 say.

Using a diagram often helps in solving a problem!



In this case, there are two 2s on the grid already, so you only need to add two more, one in each of the bottom blocks. There is already a 2 in the first column, so no more 2s can go in that column and so the only place for the 2 in the bottom left-hand block is in the square under the 4. If you find this difficult to follow, draw a line through the column and row that the given 2 is in to show that the 2s in the other blocks cannot be placed in this row or column.



Similarly, there is already a 2 in the third column, so no more 2s can go in the third column and the 2 in the bottom right-hand block must go above the 3.

You can see that the bottom row already has a 3 in it, so the 3 in the bottom left-hand block must go next to the 4. Now look at the third row, which number is missing?

As more numbers are added, the puzzle gets easier. If you have not finished the puzzle for yourself, try to do so now. Your final grid should look like this:

An important part of working on any course is the time spent thinking about what you have done and how effective your methods have been so that you can improve your work. The next activity helps you to do this.

Activity 2 How did you get on?

Spend a few minutes thinking about how you felt about doing Activity 1 and reading the comment afterwards. If you were asked to solve a similar puzzle, would you approach it differently? What have you learned from tackling the activity about your studying or how you approach problems? Jot down a few notes to summarise your thoughts.

Comment

You may have felt slightly apprehensive about being asked to do something so early in the chapter – that's a fairly natural feeling when you are not quite sure what to expect! Or you might have been keen to have a go or you might even have felt very confident if you already enjoy Sudoku puzzles. Your comments will obviously be rather personal but some things you might have noted are the hints on tackling the problem – crossing out rows and columns and concentrating on the smaller blocks or looking along the rows to see what numbers were missing. You may not have used quite the same strategy for solving the puzzle – that's fine as there is usually more than one way to solve a problem.

Activities 1 and 2 also highlight some other important aspects of working on activities. These include making notes and what you can do if you find an activity is either too easy or too hard, or if you make a mistake when tackling an activity.

Making notes

You can use sticky notes on the sides of the pages to find information or questions for your tutor easily too.

The first point is that we encouraged you to write on the book by highlighting the important information in the problem and even putting in the answer to the puzzle. The book is yours to keep, so it is fine to do this: it will help keep your learning active. You might also like to scribble on comments to remind yourself of questions you would like to ask your tutor, or to fill in extra lines of working to clarify the maths. You may like to use coloured pens to highlight different aspects of the text, for example, blue for important definitions, yellow for useful techniques, pink for sections relevant to the assignments, and so on. This will make things easier to find if you need to scan back over your work later. Some students find a traffic light system works well – red for things you do not understand, amber for areas that need more work and green if you do understand the topic and can apply it.

On the other hand, you may prefer to write your notes for the activities on paper and then store them in a folder. All your work on the activities is then kept together and you can look back over your work as a whole. Some students like to use a fairly big notebook with one page for the mathematics and the facing page for notes and comments like those you made in Activity 2. Another possibility is to take a blank piece of A4 card and write key definitions, ideas and techniques on it as you work through the chapter. Writing things down in your own words will help your understanding. You will have a handy review of the main ideas that you can refer to later, and you can also use it as a bookmark as you work through the chapter, using it to cover up the comments on the activities while you tackle them. People learn in different ways and it is important for you to experiment to find out which methods work best for you.

As you work through the course, you will meet a lot of mathematical notation and language. So that you can check meanings quickly, it is worth using a small notebook as a mathematical dictionary. Some English words, such as 'odd', 'power' or 'root', have a specific meaning in mathematics that is different from their everyday meaning. You can store these words with other mathematical definitions and abbreviations in alphabetical order in your mathematical dictionary. New symbols and mathematical notation can be stored in a section at the back. Remember to include a comment on how to read the symbol as well as its meaning. You may find it helpful to include a section for useful formulas as well.

Was the activity too hard?

Now consider how you got on with the activity itself. If you get stuck with an activity, it is fine to read through some or all of the comments under the activity and see if you can then understand how to tackle the problem. For example, in the Sudoku puzzle, the hint to cross out the column and row might have been sufficient for you to tackle the rest of the problem yourself. Or you may need to read through all the comments to fully understand what is going on. That is all right too – it is a good way to learn, and you might then like to try a similar problem from the Exercise booklet just to make sure that you have fully understood the ideas. If you are still puzzled by it, try discussing the problem with someone else or your tutor. Talking about a problem, particularly to someone who is not familiar with it, will help you to explain the problem and that might be sufficient for you to see the way forward yourself. Alternatively, a friend or your tutor might be able to suggest a different approach which will help you get started.

Using other resources

Another possibility is that you may have managed to do the particular activity, but do not feel totally confident and would like some extra practice. In this case, it is worth trying some of the examples in the Exercise booklet. You will find that comments are given for both the activities in this book and those in the Exercise booklet. Do take time to read through these as they may illustrate a different way of solving the problem and they will also show you how to set out your solutions concisely. You can practise further by using other resources. For Sudoku puzzles, newspapers are a good source, but there are also books and if you have access to the internet, there are many mathematical resources on the Web.

What if you make a mistake?

Mistakes tend to be of two different kinds, careless slips and more serious misunderstandings. If you have made a careless slip, it is worth trying to work more carefully in future and taking time to check the intermediate steps and your answer. If, however, you have misunderstood some aspect, then you might need to look back over the topic or ask your tutor for help. Sometimes you may think you do not understand something, but in reality you may just have made a slip earlier on, which then causes a problem later. So if you do find you are stuck with a problem, check back over your working first to see if you have made a mistake. Whatever has happened, the mistake will have been useful in deepening your understanding. Making mistakes is a bit like being stuck – everyone does make mistakes, but it is what they do afterwards that counts. Take the case of the eminent mathematician, Professor Andrew Wiles, who announced in June 1993 to the world's media and the mathematical community in general, in a packed lecture theatre at Cambridge University, that he had solved a famous mathematical problem called Fermat's Last Theorem. Mathematicians had been working on this

problem since the 17th century and Wiles had been working on it for seven years himself, so it was a very exciting announcement. Unfortunately, there was a mistake in the proof which took a further year to correct! Now, keeping things in perspective, the only people who are likely to know about your mistakes are yourself and possibly your tutor who is there to help you anyway. So just learn from the experience and move forward!

Was the activity too easy?

Try making your own 4 × 4 Sudoku puzzle by putting the four numbers 1, 2, 3 and 4 on a blank grid. Can you solve your puzzle?

Finally, if you are already a Sudoku expert, you will have found this particular puzzle rather easy and may have skimmed over it quite quickly. Whether or not you found it easy, as you work through the course, you will probably find topics that you do already understand and can therefore spend less time on. For a lot of activities there will be short notes in the margin (as here) that encourage you to think about a more challenging problem instead. You can also challenge yourself by asking questions, for example: what makes some Sudoku puzzles easy and some difficult? Is there only one solution? How can you tell if a starting position will give a solution at all?

There are also some more challenging activities called 'Brain stretchers'. However, if the ideas are new or you are short of time, **you can ignore these margin notes and Brain stretchers for the moment** and concentrate on the main text.

Learning from the activity

When you have worked through an activity and read the comments, it is worth thinking back on how you got on – do you need more practice? Could you have tackled the problem in a different way? How does this fit in with what you already know and your own experience? For example, are there any areas in your life where you could use the techniques that you have just learned about in a practical way? This will help you to develop a firm basis for tackling further problems confidently. You might also want to make a few notes on techniques you can use in the future either for solving problems or for improving your learning. For example, you may decide to try out one of the different ways of taking notes or remind yourself of the importance of using a diagram.

One theme of the course is using mathematics in practical situations and you may be wondering what this particular puzzle has to do with solving real problems. Well, first of all, it has introduced you to working logically and that is an important general technique that is used often. Second, Sudoku puzzles are quite similar to Latin squares which are used extensively in designing experiments, such as finding out how different crops grow with different fertiliser treatments. This happens quite often in mathematics. Very abstract mathematical topics, which seem to have no practical use whatsoever when they are discovered, may later turn out to be extremely

A 'Latin square' is a square grid of numbers in which each number occurs just once in every row and column.

important in science or technology. So just because it is difficult to see a use at the time, it does not mean that there will not be some important practical development later! When the famous physicist, Michael Faraday was asked what use his experiments with wires and magnets were, he is reputed to have replied, 'What use is a new-born baby?'. So too with pure mathematics! Mathematics is very interesting in its own right, as we hope you will discover as you work through the course.

1.2 Using the calculator

A lot of everyday problems can be solved by doing a little arithmetic and while there will be occasions when you might want to work out the calculation quickly in your head or on a piece of paper, most of the time using a calculator will be easier. However if you do use a calculator, you need to be confident that your answer is correct. This involves understanding what the different keys on your calculator do, using them appropriately and then checking that your answer is reasonable. You will find activities to help you learn how to use the course calculator in the Calculator booklet. The main text will invite you to try these activities and there will be an icon in the margin, so that when you are planning your studies, you can see where your calculator will be needed.

Work through Chapter 1 of the Calculator booklet now. This will introduce you to the layout of the calculator and how it can be used. As you work through this chapter, jot down notes to summarise what you discover about how the calculator works.

Indices is the plural of index. It is another word for powers.

As well as showing you how to use your calculator, Chapter 1 of the Calculator booklet also introduced you to the order in which more complicated calculations are carried out. To recap, you need to work out any brackets first, followed by any powers or indices. Then carry out the divisions and multiplications and finally any additions and subtractions. If part of the calculation involves only divisions and multiplications or only additions and subtractions, work through from left to right. This can be summarised by using the mnemonic **BIDMAS**, where the letters stand for **B**rackets, **I**ndices, **D**ivision, **M**ultiplication, **A**ddition and **S**ubtraction. If you would like more practice with these types of calculation, try some of the examples in Chapter 1 of the Exercise booklet.

1.3 Solving problems

People have very different feelings about maths and there are many myths surrounding the subject too. One myth is that you have to be good at arithmetic to be good at maths; and another one is that you can either do maths or you can't. Neither statement is true.

However some people do feel very anxious when faced with a problem that might involve some mathematics. This is probably why the authors of the Sudoku puzzles in the newspapers felt it necessary to explain that the puzzles used no mathematics at all, just to encourage people to try them. Fear of mathematics can create very real barriers to understanding. If your mind goes completely blank as soon as you have to deal with a calculation or problem, this can be difficult to overcome. Do not forget that this feeling of being stuck is actually very common, especially for mathematicians and it is an important part of mathematical problem solving. The difference is how people deal with these feelings. Andrew Wiles, when asked what he did when he got stuck on Fermat's Last Theorem, said:

> When I got stuck and I didn't know what to do next, I would go out for a walk. I'd often walk down by the lake. Walking has a very good effect in that you're in this state of relaxation, but at the same time you're allowing the subconscious to work on you. And often if you have one particular thing buzzing in your mind then you don't need anything to write with or any desk. I'd always have a pencil and paper ready and, if I really had an idea, I'd sit down at a bench and I'd start scribbling away.
>
> (Wiles, 2000)

Another approach is just to try and work out another way of coping – most problems can be sorted out in different ways.

Activity 3 Finding another way

The extract below illustrates how, even if someone has not got any formal mathematical skills such as measuring or even counting, it is still possible to work out other informal ways of tackling problems, and these ways do involve mathematical thinking.

> In Lesotho, a group of people discussed the different ways they use mathematics. Ntata Pitso Mafa commented that he had no formal literacy or numeracy experience. However, he can make clothes to very precise measurements (even without reading the numbers on a tape measure). He simply measures different parts of people (e.g. waists) with a string, marks it to size, and then lays the string on cloth. He calculates the amount of extra material necessary for stitching and good fit. Likewise a participant observed that he can only count to 20 but that this is not a problem – he simply counts his goats in batches of 20, dividing them into groups.
>
> (Newman, 2003)

Jot down some brief responses to the following questions.

(a) How do you think Ntata 'calculates the extra material necessary'?

(b) How many goats do you think the goat-keeper can count?

(c) Can you think of any other examples where you (or someone else) have worked out a problem in this way, using your experience but without using formal mathematical techniques?

Comment

(a) Ntata probably just estimates by eye, perhaps by allowing an extra couple of finger thumb widths on each seam (so that the material does not fray) – he will know how much to leave from experience. He will probably also add a little extra so that the material lies comfortably on the person for whom he is making the garment.

(b) If he uses one finger for each group of 20 goats, he can probably manage to count up to 200 or 10 groups of 20. But in practice, unless each group of 20 goats was grouped together in some way, counting 200 goats all running about must be fairly difficult. The number of goats counted probably depends as much on the practical and observation skills as his mathematical skills!

(c) You may have thought of many different examples where you can use an informal approach – checking bills, working out your change, estimating the time you have left before catching a bus, and so on. For example, a motorist may know from experience that when his car's petrol gauge shows it is a quarter full he can safely do another 100 miles. Try discussing this with some friends – you might be surprised at the very different ways people do tackle problems!

There are often many different ways of tackling problems, particularly real-life problems, so finding an appropriate way that you feel confident using is important. The method you use may not involve writing down a lot of formal mathematics and working out a precise answer. It may be sufficient to get a reasonable estimate that is 'good enough' for the situation. In the words of John Tukey (cited in Salsburg, 2002): 'It is better to have an approximate answer to the right question than an exact answer to the wrong one.' There will, of course, be other situations where writing down your calculations and ideas in detail will be important. We will be considering these situations later in the course.

[illegible] was a [illegible] specialised in data analysis.

The next activity involves listening to an audio track, so you will need to have a media player available as well as pen and paper. There are several audio tracks throughout the course. They have been included so that you can hear what different people feel about maths and how they use it and also to introduce you to a different way of studying.

Activity 4 Views on mathematics

Audio track 1 contains some short interviews asking people about the maths they use and how they feel about maths generally.

You may like to consider your own feelings on maths and how you use it, before you listen to the track. Some background information on the three main interviewees is given below.

The FFD (1978) requires that rivers meet standards that allow fish to live and breed.

- Mark Davis is a member of the Sussex Ouse Conservation Society. The Society is monitoring the levels of different pollutants in the River Ouse in Sussex. Mark is using maths to both support and draw attention to the Society's case that the EC Freshwater Fish Directive (FFD) for the river is not being met.

- Frazer Hollingworth has just started studying a course with the OU and wishes to improve his mathematics skills.

- Jenny Phillips works in a power station and uses maths on a regular basis.

You may find it useful to listen to the whole audio track first to get an idea of the responses. Then play it again, jotting down a few notes to remind yourself of how Mark and Jenny use maths and how Frazer feels about studying maths. Remember you can stop the track at intervals, perhaps after each interview, to help you to do this. You will meet the Conservation Society again later in the course, so making a few notes now will provide a useful reminder of the issues without having to listen to the audio section again.

Comment

Your response to this activity will be rather personal.

Mark uses maths to support the work of the Sussex Ouse Conservation Society. This involves collecting regular samples of the river water and measuring the levels of different pollutants. These measurements are then used to calculate average values over the year to see if the water complies with the Freshwater Fish Directive. He also uses maths to present the information on the website in a graphical form that people can understand quickly and easily. He is using maths in his spare time to support and publicise a cause that he feels passionately about.

Frazer has just started studying an OU maths course and it's already having an impact – he feels more confident about tackling mathematical problems and has developed a range of strategies to cope, from talking to his tutor or friends to just taking a break and coming back to the problem later. He uses maths every day, both at work and at home.

Jenny uses lots of calculations in her job and has to react quickly to changes in the electricity supply and demand to ensure that everyone gets the electricity they want, when they need it. This involves working with very large numbers and working out calculations quickly either in her head or with a calculator.

Did anything surprise you about the interviews? Have the interviews changed your view about tackling or using mathematics?

How do you think a research mathematician would describe what it is like to do mathematics? Well here is one view, from Andrew Wiles again.

> Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of – and couldn't exist without – the many months of stumbling around in the dark that precede them.
>
> (Wiles, 2000)

Studying a maths course, where there are materials to guide you and people to help, is different from tackling your own research. Although we do expect you to get stuck at times, we also expect you to use your time effectively. So if you have made a determined effort to sort out a problem, have tried some of the problem-solving strategies and are still stuck, it is better to contact

your tutor for help so that you can carry on with the course as planned, rather than continuing to try to puzzle it out. We are not expecting you to spend months in the dark! Sometimes a quick hint or a few minutes explanation will be all that you need to get yourself going again anyway. We hope that these extracts will help to convince you that learning mathematics really does involve making mistakes and getting stuck, but that there are ways to help you overcome these problems and develop a greater understanding after all.

1.4 Developing your study skills

In Section 1.1 we discussed ways of tackling activities and you have now had a go at several different types – doing a mathematical puzzle, exploring some calculator activities, reading a very short extract and listening to an audio track. These activities will form an important part of your learning but there are some other skills to develop as well. These are:

- academic reading skills
- writing skills
- planning the most effective use of your study time.

This section describes these skills and you will develop them further as you progress through the course.

Academic reading skills

In order to make the best use of your study time, it is important to have a range of reading skills. Although most people feel fairly confident reading steadily through a chapter from start to finish, there will be situations where it is more important to read through quickly to identify a key bit of information or to skim-read over several sections to get an overall view of the main ideas. In these cases, reading every sentence carefully would not be appropriate and would waste quite a lot of time. Suppose you wanted to find one bit of information quickly – for example how to tackle a particular calculation or some fact that you knew you had read about in the current chapter. What would you do?

One way would be to check in the index which would give you the page reference and allow you to home in on the information fairly quickly. Alternatively, if the topic was not in the index, you could look at the contents page or flick quickly through the chapters, skim-reading the headings and subheadings. The first sentence in each paragraph often introduces the main idea too. Or if you have a good idea where you read the information initially, you can glance quickly over each page just looking for that one piece of information.

From the course so far, try to find the following bits of information, quickly:

- suggestions for ways to make notes
- what a Latin Square is
- the main mathematical topics that you will be studying in the course
- the name of the person who was interviewed from the Sussex Ouse Conservation Society.

Comment

Skim-reading through the headings and subheadings identifies the 'Making notes' paragraph in Section 1.1. The index gives the page number in Section 1.1 for Latin squares, and the details are in the margin note on that page. The Contents page lists the main mathematical topics or you can skim over the different headings in the chapters. Skim reading over Activity 4 'Views on mathematics' quickly shows that the person who was interviewed is Mark Davis.

From this activity, you can probably appreciate why it is useful to write on the text or highlight key words and phrases as you work through, as it may help you to find important parts quickly later.

Skim-reading can also be used when you want to get a quick overview; for example, when you are planning your week's studying. This involves looking quickly through headings, subheadings, first lines of paragraphs, introductions and conclusions. At the end of each chapter, there is also a box entitled 'Study checklist' and this gives you a good idea of what has been covered too. If you do this before starting a new chapter, you will have a good idea of what is coming up, how familiar you are with it and what you should have learned by the end of the chapter. This can help you to make sensible notes because you have more idea of where the chapter is going. It can also help you to decide how long you might need to spend on the chapter and hence plan and organise your time accordingly. For example, 'I know how to round numbers so that section will probably only take half an hour, but I have always struggled with negative numbers so I'll probably need three or four hours there' or 'There's a calculator section there, so I can take my calculator into work and go through that in my lunch hour'. Skimming back over the chapter is also useful to remind yourself of what you have studied when you are reviewing your work.

Activity [illegible] Looking ahead to Chapter 2

Skim read quickly through Chapter 2 to get an overview of what is in it and how familiar you are with the topics.

Comment

Here is what one student said.

> A quick look at Chapter 2 tells me that it is about numbers starting off with an audio track on the history – will need media player for that! I'll need to review some basic skills like multiplication but there seem to be some puzzles and activities to help with these. That shouldn't take too long. I'm OK with metric units but not how to convert from one to another. Then there's work on rounding and negative numbers which is all new to me and it looks like the maths is getting serious there. There's a calculator section and some advice on reading maths as well, so quite a 'meaty' chapter overall.

Your experience will probably be different from this!

Of course, when you start to study the chapter in earnest you will have to read it through carefully and actively too, scribbling questions and notes in the margins. In Activity 3 'Finding another way', we asked you some questions, but these are just the sorts of questions you can ask yourself as you read through the text. For example, useful questions are: why is this true? What would happen if ...? What does that mean? How does this fit in with what I already know? Asking questions does help to keep your learning active.

Sections that will need very slow and careful reading are those that involve mathematical arguments and use mathematical notation. Here every symbol and word counts, so you do have to study every single part to make sense of what you have read and that takes time. We look at this in more detail in Chapter 2, but you can get a feel for mathematical arguments by looking back at the comment about tackling the Sudoku puzzle in Section 1.1, particularly the explanation of where the 2s go in the bottom blocks. If you were unfamiliar with this puzzle, you may have had to read this several times and perhaps draw a diagram or work it out some other way to convince yourself what these statements meant and that they were correct.

Writing skills

During the course, you will be writing for different purposes. These include responding to the activities, making your own notes (as mentioned in Section 1.1), writing your assignments and explaining mathematical ideas. These forms of writing use different styles. For example, what might be perfectly acceptable as a quick note to remind yourself of a key point might not be suitable as part of your assignment for your tutor to read. You will need to make sure that whatever you write is going to be clear to anyone who needs

to read it and that includes yourself in several weeks. Asking someone else to read your work or imagining that you are writing for someone who has little knowledge of the topic and requires a full explanation can help to make sure you include sufficient detail. Mathematical writing requires additional skills in using notation and specialist vocabulary, as it does use its own language. This skill does take time to develop. You will be learning more about writing good mathematical arguments in Chapter 3. Meanwhile as you work through the next few chapters, observe how the mathematics is set out so that it is clear to read.

Planning your study time

One of the most difficult aspects of being an adult student is fitting in your studying with everything else in your life. So it is important both to find enough time to study and then to try to make the most effective use of your time. Finding enough time can be quite a challenge! It often means giving up some activities you currently enjoy or perhaps negotiating with your family and friends to pass on some of the daily chores or to allow you some time to yourself. It is surprising, though, how much can be achieved in a 10 minute slot such as recapping on previous work, sorting out paperwork or administrative tasks, planning future work or working through an example or two. Having found some time, it is also worth thinking about whether this is the best time for you to study and, if it is not, changing it.

Activity 7 Planning your study of Chapter 2

Based on your preview of Chapter 2 in Activity 6 and also your experience of studying this chapter, try to plan the times for your study sessions for Chapter 2. Remember to include some emergency time in case your timetable does not work out as planned or in case you want to include further work on this chapter.

Comment

The student mentioned in Activity 6 made the following comment.

> I think I'll allocate four study sessions to Chapter 2. I work best in my bedroom in the evenings – that's the quietest place and I can concentrate without interruptions. Monday and Thursday evenings fit in best with other commitments so this chapter will take about two weeks. I'll keep a Saturday morning free as well, just in case my timetable slips and I'll fit in the calculator work in odd half-hour sessions during the week – probably going on the train into work.

It is worth considering the times of day and the lengths of sessions where you work most productively. For example if you know you are going to lose concentration after half an hour or so and also that you are just too tired to study in the evenings, it is probably better to schedule your study time for new topics in half-hour slots in the morning and use the evenings for other chores.

1.5 Conclusion

This chapter has introduced you to the main learning components of the course. It also outlined the main themes of developing your mathematical skills, the strategies you can use to cope with difficult problems and more general study skills. However, the key question is what have you learned? What do you now know, or can do, that you did not know, or could not do, at the start of the chapter? When you reach the end of a chapter it is a good idea to review the work you have completed. Activity 8 helps you to do that and it is a technique you can use for other chapters too.

Activity 8 Looking back

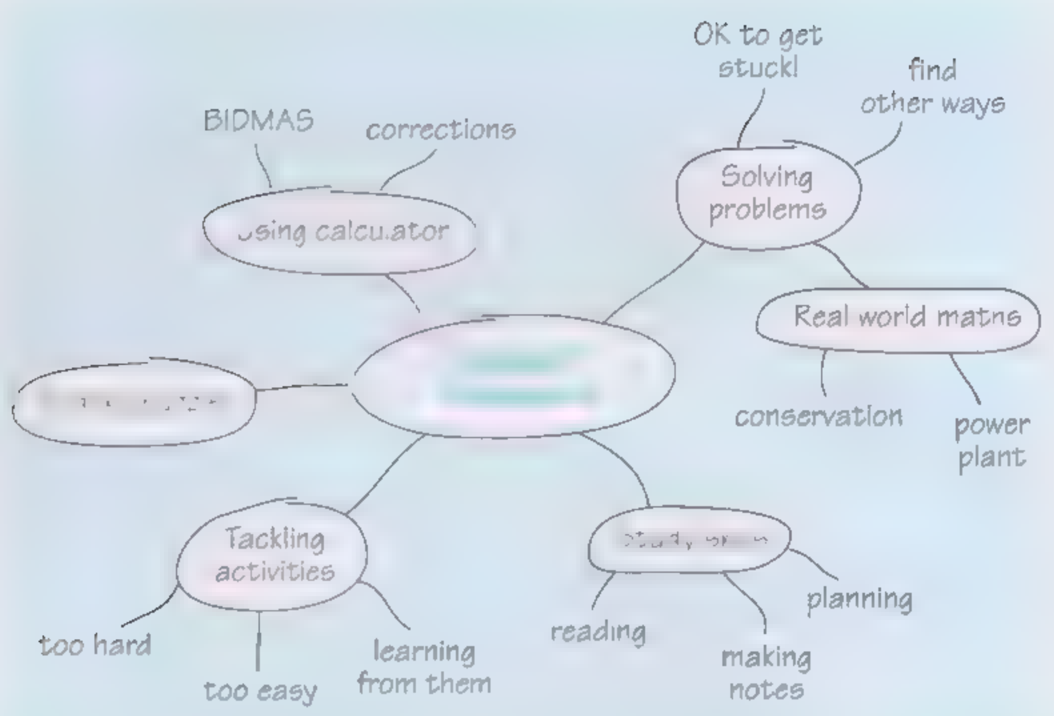
Summarise what you have studied and then check your progress against the study checklist below. If you tick off the activities you can do, you will be able to see clearly which areas to ask for extra help on or to practise further. You may find it helpful to look back over your notes and activities. If you were using the traffic light system, how many red or amber sections are still outstanding? You will then be able to build sessions for these aspects into your timetable for future study sessions, ensuring a firm basis for your later work. This is particularly important for mathematical skills, because later skills often depend on earlier understanding. Do you need to adjust your plan for Chapter 2 or recap on any of the skills in this chapter?

Comment

To quickly summarise a chapter, some people find it helpful to draw a 'spray diagram' (also called a 'spider diagram') which shows how the different topics are connected together.

Start by writing down the central theme and then work outwards. The process of creating the diagram can help to sort out ideas and how they fit together. These diagrams are very personal and can be as elaborate or as simple as you wish. Some students like to add a lot of detail, for example including colour, pictures, page references and examples, and others prefer a simple plan, concentrating on the key points. An example for this chapter is shown opposite.

Do not worry if your diagram looks different from this one. It is your personal record of the content of the chapter and how the different sections relate. You may find it helpful to create your own spray diagram for each chapter and use it to review your work at a later date.



We hope that you will enjoy your mathematical journey through the rest of the course!

Study checklist

You should now be able to:

- tackle activities with more confidence
- use your calculator for the four operations of +, −, × and ÷, brackets and powers and also know how to correct mistakes
- carry out calculations in the correct order using BIDMAS
- appreciate that people use different strategies to help solve problems
- appreciate that maths can be used in many different situations
- start to make some useful notes
- begin reading effectively using different techniques and speeds for different purposes
- start reviewing your learning and planning further studying.

2 Numbers, numbers everywhere

This chapter is concerned with numbers and the ways that they are used to solve problems in everyday life. It introduces some important techniques for working with numbers, such as deciding what calculations are needed, estimating an answer either in your head or on paper, using your calculator and understanding the measurement system used in maths, science and technology. You will also see the tactics you can use when reading and getting to grips with some mathematics. Finally the chapter gives a taste of two important aspects of mathematics – the idea of proof and the mathematical modelling cycle which can be used to help to solve many real-life problems.

Think about an everyday task such as planning a journey. Just for a moment, imagine a world without any numbers at all and think how you would complete the task. How would you indicate the time to set off, the distance, how fast you were travelling or the cost of the journey? Numbers whether you are using them for counting, measuring, identifying or calculating are essential for solving a huge variety of problems. Even though it is difficult to imagine living without numbers now, the number system and the notation we use today have taken centuries to develop.

This chapter begins with a brief history of this process, from the Ancient Egyptians who managed to build the pyramids by using simple fractions and just doubling and halving numbers, to the Indians and Arabs who developed the number system that spread to Europe in the 13th century, and that we still use. This development is still continuing today. One of the main reasons the number system has developed is so that people can use numbers to solve a wide variety of important problems in trading, building and navigating, although we will be concentrating on more everyday problems in this chapter. As you work through the chapter, do try out some of the ideas mentioned in Chapter 1 for learning actively.

2.1 Some history

Activity 8 A brief history of numbers

Listen to audio track 2, which is about the history of number. You will hear two members of the course team, Alice and Hilary, discussing the development of the number system.

There may be some ideas, such as negative numbers, that you have not met before. Don't worry about this now – you can always return to the audio later.

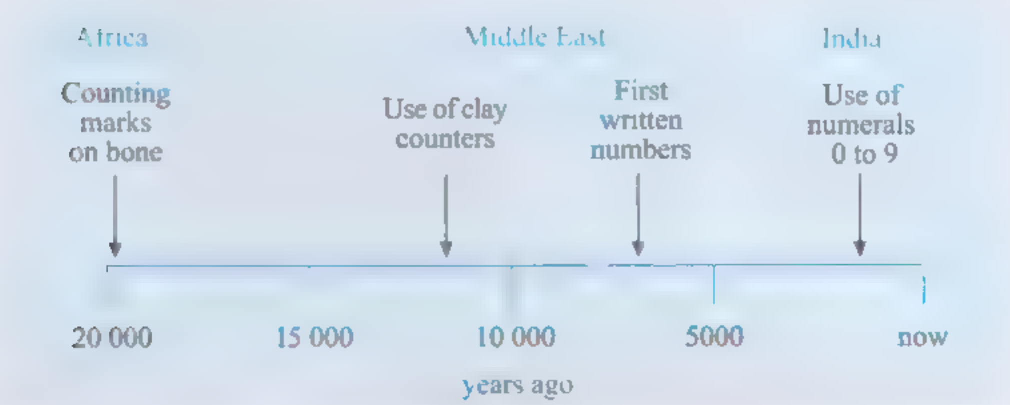
(a) Make a few notes to summarise how the number system developed including important places and dates (you may like to stop the track at intervals).

(b) About 1500 years ago, in an Indian book on cosmology called *The Parts of the Universe*, the number fourteen million, two hundred and thirty-six thousand, seven hundred and thirteen was written down. How would you write this number today?

(c) How would you write a billion using the decimal system?

Comment

(a) The main ideas, dates and places mentioned in the audio are shown on the time-line below.



The Rhind papyrus dates from just over 3500 years ago and Fibonacci wrote his book *Liber Abaci* in 1202. Mathematical developments, including other place-value systems, have also taken place in China and South America. The history of how mathematics has developed over the centuries is a long and fascinating one which involves many different cultures.

(b) One of the major breakthroughs in the history of numbers was the introduction of zero and its use in the decimal number system which we use today. This system uses ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9), and the position of each digit in the number indicates its value. For example the number 402 represents 4 hundreds, 0 tens and 2 units. The table below shows a place value table and shows the number 'fourteen million, two hundred and thirty-six thousand, seven hundred and thirteen' written in this form.

Place-value table

Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Units
10 000 000	1 000 000	100 000	10 000	1000	100	10	1
1	4	2	3	6	7	1	3

This number is written 14 236 713. Notice how leaving a small gap between each group of three digits makes the number easier to read.

In a decimal place value table, the value of each column is ten times the value of the column to its right.

You can extend the decimal system indefinitely for larger whole numbers by adding extra columns on the left and labelling the columns as hundred millions, thousand millions, ten thousand millions, and so on.

(c) In the UK, a billion is now taken to mean a thousand million and can be written as 1 000 000 000.

Using a number line

A number line can help you to visualise many different kinds of numbers as you saw on the time-line above. For example, on the number line below, the intervals between the whole numbers (or the units) have each been split into ten equal intervals – these are tenths. If each tenth is then split into ten equal intervals, each of these smaller intervals will be hundredths since there will be $10 \times 10 = 100$ of these intervals in a whole unit. The number line shows the numbers, 'two-tenths', 'one unit, three-tenths and five-hundredths', and 'one unit and eight-tenths'.

To write these numbers in decimal form, the place-value table can be extended by adding columns to the right, as shown below.

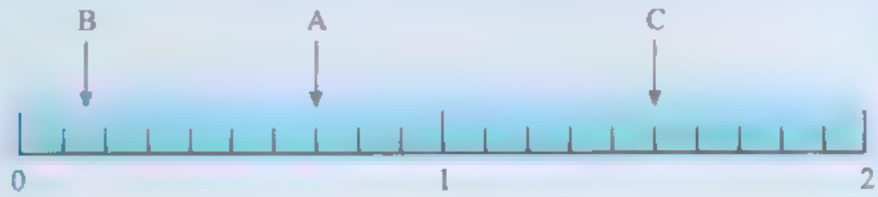
Units	•	Tenths $\frac{1}{10}$	Hundredths $\frac{1}{100}$	Thousandths $\frac{1}{1000}$
	•	2		
	•	3	5	
1	•	8		
4	•	2	0	5

Since the value of each column is ten times smaller than the value of the column to its left, the columns to the right of the units column will represent 'tenths', then 'hundredths', 'thousandths', and so on.

To separate the decimal fractions from the whole numbers, a 'decimal point' is used. (In mainland Europe a comma is used, for example €1,35.) So 'two-tenths' is written 0.2 and 'one unit, three tenths and five hundredths' as 1.35 – note that this last number is read as 'one point three, five', **not** 'one point thirty five'. Similarly the number 4.205 is read as 'four point two nought five'. If the number does not have a whole number part, a zero is written in the units column (e.g. 0.2 or 'nought point two'). This makes the decimal fraction easier to read – the decimal point in .2 could easily be missed and the number misread as 2 otherwise.

Activity 10 Numbers on the line

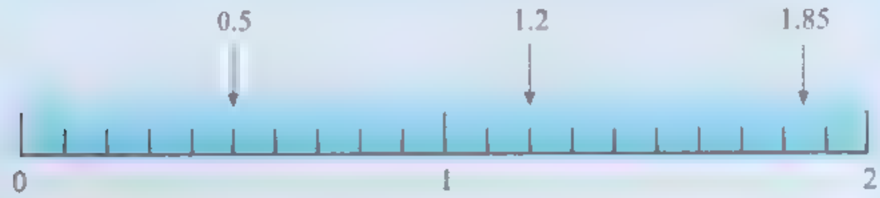
(a) Mark the numbers 0.5, 1.2 and 1.85, on the number line below.



(b) What numbers are indicated by A, B and C? When do you use decimals in everyday life?

Comment

(a) The numbers 0.5, 1.2 and 1.85 are indicated on the number line below.



(b) A marks the number 0.7, B marks the number 0.15 and C marks the number 1.5. You might meet decimals in your everyday life when you are shopping or measuring something.

Activity 11 Which is tallest?

The final of a sunflower growing competition has identified the three tallest sunflowers grown by three different people (Ahmed, Bert and Cathy). The heights of the sunflowers have been measured. Ahmed's is 1.8 metres, Bert's is 1.67 metres and Cathy's is 1.72 metres. Which is the tallest sunflower and which is the smallest?

Comment

To decide which of these three numbers is largest, you can either draw a number line and mark the heights on it or use a place value table as shown below.

Sunflower	Units 1	•	Tenths $\frac{1}{10}$	Hundredths $\frac{1}{100}$
Ahmed	1	•	8	0
Bert	1	•	6	7
Cathy	1	•	7	2

Starting at the left-hand column, compare the digits in each column. As all the digits in the units column are 1, look at the tenths column. The first number has eight tenths, the second number has six tenths and the final number has seven tenths. So the first number is biggest, followed by the third number and the second number is smallest. This means that Ahmed has grown the tallest sunflower with a height of 1.8 metres and the smallest sunflower was grown by Bert with a height of 1.67 metres.

In order to compare the heights of the sunflowers, careful measurements were needed and, in this case, measurements were rounded to the nearest centimetre. In practice, the precision of any measurement depends on the physical situation as well as the measuring instruments used. The next section shows you how to round any number.

2.2 Rounding

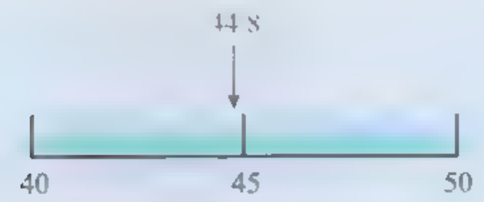
In July 2005, the population of the world was estimated to be 6.4 billion people. Clearly that is not an exact value as people are being born (or dying) every second, but it does give you an indication of the world's population at that particular time. In newspapers and elsewhere, numbers are often rounded so that people can get a rough idea of the size, without getting too bogged down in the detail. You may already use rounding in your everyday life, perhaps rounding prices in a supermarket (£2.99 – that's about £3) or distances (it's about 20 miles away) or times (it took about 40 minutes to get there).

Using approximate values is also very useful if you want to get a rough idea of the answer to a calculation before using your calculator. This acts as a check on your calculation and may help you to spot if you have entered a wrong number or an incorrect key sequence. In this section, you are going to look at rounding in more detail and then, in the next section, you will see how to use it to estimate the answers to various problems.

Suppose you are planning a journey by road from Worcester to Birmingham. According to a website travel planner, the distance is 44.8 kilometres (km). How far is it approximately?

44.8 km is between 44 and 45 km and, from the number line, you can see that it is closer to 45 km. So to the nearest kilometre, the distance is 45 km.

Alternatively, the distance can be rounded to the nearest 10 km: 44.8 lies between 40 and 50 and is closer to 40. So, rounding to the nearest ten kilometres, Birmingham is about 40 km from Worcester.



From the number line, you can see that numbers bigger than 45 will round up to 50 and numbers smaller than 45 will round down to 40. However, 45 is just as close to 40 as it is to 50; by convention, it is rounded up.

Activity 12 How far is it approximately?

Suppose you were planning a journey by road from Dover to London (73 miles), London to Manchester (185 miles) and then Manchester to Newcastle (136 miles). Round each of these distances to the nearest 10 miles.

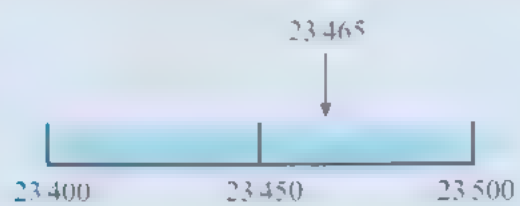
Comment

As 73 is between 70 and 80 and is closer to 70, it rounds down to 70. The distance between Dover and London is 70 miles, to the nearest 10 miles.

As 185 is exactly halfway between 180 and 190, it rounds up to 190. The distance between London and Manchester is 190 miles, to the nearest 10 miles.

As 136 is between 130 and 140 and is closer to 140, it rounds up to 140. The distance between Manchester and Newcastle is approximately 140 miles, to the nearest 10 miles.

In some situations, it may be more appropriate to round numbers to the nearest hundred or the nearest thousand or some other value. For example, suppose the number of supporters attending a football match was 23 465.



To round this number to the nearest 100, decide which two 'hundreds' the number lies between and choose the closer one. Here, 23 465 lies between 23 400 and 23 500. Halfway between these two numbers is 23 450 and since 23 465 is greater than this halfway number, it rounds up to 23 500.

The same answer can be obtained by finding the 'hundreds' digit in the number (in this case, 4) and then looking at the digit to its right (in this case 6). If it is 5 or more, round up. If it is 4 or less, round down. So 23 465 rounds up to 23 500.

hundreds digit

is this digit 5 or more?

rounded value

$$23\,465 \approx 23\,500$$

approximately equals sign

'Approximately equals' may also be written using the ≅ symbol.

Similarly, to round the number to the nearest thousand, look at the digit in the thousands column – in this case, 3. So the number lies between 23 000 and 24 000. The next digit on the right is 4. This is less than 5, so the number rounds down to 23 000.

Activity 13 Rounding headlines

The following numbers are to be used in newspaper headlines. Round the numbers as indicated below.

(a) 2475 car workers lose jobs (nearest 100).

(b) 165 people protest at school closure (nearest 10).

(c) Student wins £35 321 (nearest 1000).

Comment

(a) As 2475 is between 2400 and 2500 and is closer to 2500, it is rounded up to 2500. So 2475 ≈ 2500, and the headline becomes '2500 car workers lose jobs'.

(b) As 165 is halfway between 160 and 170, it is rounded up to 170. So 165 ≈ 170 and the headline becomes '170 people protest at school closure'.

(c) As 35 321 is between 35 000 and 36 000 and is closer to 35 000, it is rounded down to 35 000. So 35 321 ≈ 35 000 and the headline becomes 'Student wins £35 000'.

In practice, you will need to decide whether to round numbers to the nearest unit, ten, hundred or thousand and this will depend on the situation and how accurate the result needs to be.

For example, when out shopping it is easy to get carried away and spend more than you intend. This is particularly true if you do not carry much cash but rely on debit cards or credit cards to make purchases. The cost of shopping can come as less of a shock if you start to estimate how much you are spending as you collect the goods. This is what Jenny said about estimating her shopping bill.

> I've stopped paying by plastic because I was spending too much. Now I carry cash and I know how much I've got to spend. What I do is guess at the total by doing a sort of running total in my head. If it's a price that's sort of in between then I go up to the nearest pound. Once I went down to the nearest pound with some things and didn't have enough money when I got to the check-out.

The degree of accuracy will always depend on the situation you are in. When it comes to DIY it can be very important to be accurate because you might buy too much or too little of an item. Or you might buy a new bath and discover that it cannot fit into your bathroom. These can be very expensive mistakes.

2.3 [illegible] and subtraction

One of the most important advantages of the decimal number system is that calculations can be carried out easily. This section briefly considers the fundamental operations of addition (+) and subtraction (−) to revise these concepts and show how they can be used in tackling problems. Although you will be using your calculator for a lot of calculations, there will be occasions when it is useful either to work out an answer to a calculation on paper or carry out a quick calculation in your head.

Section 2.4 covers multiplication (×) and division (÷).

Addition

If you are working out a budget, checking a bill, claiming a benefit or work expenses, working out the distance between two places by road or many other everyday calculations, you will probably need to add some numbers together. Consider the following questions.

- How much is the bill **altogether**?
- How much does the holiday cost now that the price has **increased** by £50 due to fuel surcharges?
- Five **more** people want to come on the trip – what is the **total** number of people booked on the coach now?
- What is the cost **plus** VAT?

All the questions involve addition to work out the answer. The process of adding numbers together may also be referred to more formally as finding the **sum** of a set of numbers.

One way of doing this is shown below.

The + and – symbols for addition and subtraction became widely used in England after the publication of The Whetstone of Witte by Robert Recorde in 1557.

Adding numbers

For example 26.2 + 408.75 + 0.07

Write the numbers underneath each other so that the decimal points and the corresponding columns line up. Then add the numbers in each column starting from the right.

```
 26.20
408.75
  0.07 +
435.02
```

Activity 14 Checking a bill

Suppose you buy three items costing £24.99, £16.99 and £37.25 from a mail-order catalogue and the postage is £3.50. Round these prices to the nearest pound and work out an estimate for the total bill either on paper or in your head. Use your calculator to work out the total and check that this total is close to the estimate.

Comment

The four rounded prices are £25, £17, £37 and £4. To find an estimate for the total cost, add the four rounded prices together. The answer is 83, so the total bill will be approximately £83.

```
25
17
37
 4 +
83
```

Alternatively, you can work this out in your head. This is easier to do if you split each number, apart from the first one, into tens and units, so 17 can be split into 10 + 7 and 37 into 30 + 7. The sum then becomes 25 + 10 + 7 + 30 + 7 + 4. Then working from the left and adding each number in turn, you can say '25 and another 10 makes 35; another 7 gives 42; another 30 gives 72, 7 more makes 79 and another 4 gives 83'.

You may have your own method of working things out in your head – that's fine.

Working out the total on your calculator gives an answer of £82.73. This is close to the estimated value of £83, so you can be fairly confident that it is correct. Notice how breaking the problem into a series of easy steps made it easier to sort out.

For extra practice, go back to Activity 12 and using the rounded values, work out an estimate for the total distance for the route from Dover to Newcastle. Then use your calculator to work out the distance from the numbers given in the question.

You should find that the estimate (400 miles) is reasonably close to the calculated value (394 miles).

The next activity is a number puzzle, which introduces some more ways of adding numbers quickly in your head – you will see how to turn it into a party trick to amuse your friends after Activity 16!

Activity 15 A party puzzle

Write down the numbers 1 to 25 in order, in a 5 × 5 grid, so that the first row reads 1, 2, 3, 4, 5, and so on. Now choose five numbers from the grid as follows.

- For the first number, choose any number from the grid, say 17.
- Now cross out the numbers in the same row and column as the chosen number, 17.
- Choose the second number from the remaining numbers on the grid, say 24. Then cross out the numbers in the same row and column as the second number.

1	~~2~~	3	4	5
6	~~7~~	8	9	10
11	~~12~~	13	14	15
~~16~~	~~17~~	~~18~~	~~19~~	~~20~~
21	~~22~~	23	24	25

Continue in this way until you have chosen five numbers. For example, I chose 17, 24, 15, 6 and 3.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Now add the five numbers together in your head or using pencil and paper – you can check your answer with your calculator if you like.
Here, the sum is 17 + 24 + 15 + 6 + 3 = 65.

Try the puzzle choosing different numbers, at least three more times using your calculator, working out the sum on paper or doing the calculation in your head. What do you notice about the different sums?

Comment

Whichever numbers you choose, you should find that the sum is always the same, 65.

> Can you work out how to prove this?
>
> Hint: Write each number as a sum of the number in the top row of the column plus another number e.g. 19 = 4 + 15.

By specifying the way that the numbers must be chosen, exactly one number is chosen from each row and each column of the grid. It is then possible to prove that the sum will always be 65, without trying all the different choices of five numbers. In particular, you can choose the numbers on the diagonal from top left to bottom right (the leading diagonal) because this gives one number in each row and column. The sum is then 1 + 7 + 13 + 19 + 25 = 65. Finding the total of the diagonal elements is a quick way of working out the sum for this puzzle.

You can work out the sum in your head by breaking the numbers down into tens and units as before. Similarly if you find it difficult to add on 6, 7, 8 or 9, break down the problem by adding on 5 and then 1, 2, 3 or 4, respectively. When you are adding two numbers together, it does not matter which order you add them together, so $2 + 3$ gives the same answer as $3 + 2$. Mathematically, we say that addition is **commutative**. This means that you can also look for numbers which add together to give 10, 20, and so on. Here, if we rearrange the sum to $(7 + 13) + (1 + 19) + 25$ and work out the sums in the brackets first, we get $20 + 20 + 25$, which is 65 as before. Another way to add 13 to 19 is to say, '19 is 1 less than 20 and 13 plus 20 is 33, so 1 less gives the answer as 32.' People do carry out these mental calculations in many different ways, and several involve splitting up the calculation into easier steps. The details of your methods do not matter as long as you can find a way that you can use easily and confidently to get the right answer.

It can be useful to be able to work out some things in your head quickly, both in everyday situations and also if you want to get a rough idea of the answer to a calculation before using your calculator. So these techniques are worth practising.

Activity 16 All in your head

Try working out the following problems in your head using the strategies above.

(a) $6 + 17$

(b) $25 + 13$

(c) $29 + 19$

(d) $12 + 5 + 8 + 3 + 2$

Comment

This is the way I tackled these. Your method may be different.

(a) $6 + 17 = 17 + 6 = 17 + 3 + 3 = 20 + 3 = 23$.

(b) $25 + 13 = 25 + 10 + 3 = 35 + 3 = 38$.

(c) Here I would say that 29 is one less than 30 and 19 is one less than 20, so $29 + 19$ is two less than $30 + 20$ or 50. So the answer is 48. Alternatively, you could write $29 + 19 = 30 - 1 + 20 - 1 = 50 - 2 = 48$.

(d) $12 + 5 + 8 + 3 + 2 = (12 + 8) + (5 + 3 + 2) = 20 + 10 = 30$.

You can practise these mental techniques in lots of different ways but it is a good idea to start with some situations that you feel comfortable with initially. For example, you could just work out the cost of two or three small items priced in pence, such as a newspaper at 90p plus some chocolate at 35p. As you become more confident, you can always progress to more complicated problems. Practising maths in everyday situations like this can help your fluency, confidence and skill.

Now for the party trick! Write down the number 65 on a piece of paper and fold it up so that the number is not visible. Announce that you are going to predict the sum of the five numbers your friend chooses from the grid. Ask the friend to choose the numbers following the rules given above and also to work out the sum. Assuming no mistakes have been made, you can then dazzle your audience by producing the piece of paper with exactly the same sum written on it! It is not a good idea to repeat the 5 × 5 puzzle with the same audience though, otherwise your technique may be rumbled! However you can extend these grids or even invent some new puzzles of your own.

Activity 11 Optional Brain stretcher: Bigger puzzles

start with 1?

For example, 20, 21, 22 .....

Or go up in 2s: 2, 4,

Draw out the grids for a 6 × 6 and a 7 × 7 puzzle. For each grid, work out the sum of the numbers on the diagonal. Use some of the mental strategies if you like. Try the puzzles out with these different grids. Do your predictions for the total of the chosen numbers work?

Comment

If you draw a 6 × 6 grid, the numbers on the diagonal are
1 + 8 + 15 + 22 + 29 + 36 = 111 and the sum for the 6 × 6 grid is also 111.

Similarly the sum for the 7 × 7 grid is 1 + 9 + 17 + 25 + 33 + 41 + 49 = 175.

If you decide to show these different puzzles to someone else, you can either try to remember the different sums or quickly work out the sum of the numbers on the leading diagonal in your head, as you display the grid.

Subtraction

Consider the following questions.

- What's the **difference** in distance between going that way and this?
- How much **more** money do we need to save?
- If we **take away** 45 of the plants for the front garden, how many will be left for the back garden?
- If all holiday prices have been **decreased** (or **reduced**) by £20, how much is this one?

All these questions involve the process of subtraction to find the answer – a process that you often meet when dealing with money. For example, if a shopping bill is £7.85 and you pay with a £10 note, your change is what's left when £7.85 has been taken from £10. Breaking the problem down into stages, taking £7 from £10 leaves £3. Then taking away 80p leaves £2.20 and then taking away another 5p leaves £2.15.

Alternatively, you can think of subtraction in another way as undoing the process of addition. For example, instead of saying '£10 take away £7.85 leaves what?' you could say 'What do I have to add to £7.85 to get to £10?'.

Then adding on 5p gives £7.90, another 10p gives £8 and another £2 will give £10. So the amount to add on is £2.15 altogether. This is the same answer as the one obtained by subtraction.

Working out calculations like this on an everyday basis may help you to feel more confident in working with numbers. You can always check your answer on your calculator, if you like.

Now imagine a slightly different problem. Suppose you have £25.90 in cash (two £10-notes, five £1-coins and nine 10p-coins) and you owe a friend £16.87. In order to give your friend the exact money, you would need to change some of your money. For example, if you exchanged one 10p-coin for ten 1p-coins, you would be left with eight 10p-coins and ten 1p-coins. You could then give your friend 87p as eight 10p-coins and seven 1p-coins. Similarly, if you exchanged one £10-note for ten £1-coins, you would then have fifteen £1-coins and could give your friend six of them together with the remaining £10-note. You would then be left with nine £1-coins and three 1p coins or £9.03. This illustrates a general method of subtraction, as shown in the box below.

Subtracting one number from another

For example 25.90 – 16.87

Set out the calculation so that the decimal points and the corresponding columns line up. Then subtract the bottom number from the top number in each column, starting at the right, and changing the amounts in each column as needed.

change 2 tens
into 1 ten and 10 units
giving 15 units in all

change 9 tenths
into 8 tenths
and 10 hundredths

$$\begin{array}{r} {}^{1}\,{}^{1}\;\;{}^{8}\,{}^{1} \\ \cancel{2}5.\cancel{9}0 \\ 16.87\;- \\ \hline 9.03 \end{array}$$

You may already be familiar with other ways of subtracting numbers. This is fine; just check that you can use either this method or your own confidently. Note that you can check your answer (9.03) by adding it to the number you subtracted (16.87) – you should end up with the number you started with (25.90). Consider whether you need to practise these skills further either by applying them to real-life situations or by making up some examples of your own and checking the answers on your calculator.

Activity 18 Train journeys

To get to a meeting, I can travel by train leaving at 9.35 a.m. and arriving at 11.10 a.m. The ticket costs £48.30 but I have a refund voucher of £15.75 following the cancellation of a train on an earlier journey. How long will the journey take and how much money will I have to pay for the ticket if I use the refund?

Comment

Working with times can be tricky because you have to remember that there are 60 minutes in an hour, so you are not working with the simple decimal system. Imagine a clock and say: 'From 9.35 to 10.00 is 25 minutes; 10.00 to 11.00 is an hour and 11.00 to 11.10 is an extra 10 minutes. So the journey time is 1 hour and 35 minutes, assuming the train runs on time!'

To find out how much I will have to pay for the ticket, I need to subtract £15.75 from £48.30. Doing a formal subtraction in my head isn't easy, so I'll work up from £15.75. Adding on 25p gives £16, then I need £32 to get to £48 and another 30p to reach £48.30. So the total I have to pay is £32 plus 25p plus 30p or £32.55.

Activity 19 A surprising result?

The table below gives a list of instructions for a number puzzle and an example to illustrate it. Work through the instructions yourself three times, putting your answers in the last three columns of the table. What do you notice?

The 1089 puzzle

Instruction	Example	Try 1	Try 2	Try 3
Write down a three-digit number in which the first and last digits differ by 2 or more	417			
Reverse the digits	714			
Subtract the smaller number from the larger	714 − 417 = 297			
Reverse the digits of the answer	792			
And add to the previous answer	792 + 297 = 1089			

What happens if you choose a number in which the first and last digit are either the same (such as 252) or differ by one (such as 435)? Can you explain why you get these answers?

Comment

For the original puzzle, you should find that you get 1089 every time.

If the first and last digits are the same, when you reverse the number, you end up with the same number again. So subtracting one from the other gives zero. Then adding zero to zero results in a final answer of zero as well.

If the first and last digits differ by one, it looks as if the sum will always be 198. But why is this true? Can you prove that you will always get 198 if the first and last digits differ by one? You can tackle this problem by considering the hundreds, tens and units in turn. Imagine the number and the reversed number. The hundreds digit of the larger number will be one more than the hundreds digit of the smaller number, the tens digits will be the same and the units digit of the larger number will be one less than the units digit of the smaller number. So if we subtract the hundreds first and then the tens, one hundred will be left. As there is one more unit in the smaller number, this will have to be taken away from this hundred, leaving a final answer of 99. Reversing this number and adding it gives 198. It is possible to use a similar argument if the first and last digits differ by 2 or 3 or 9 to show that the answer is always 1089. A proof like this in which you consider all the different possibilities, is known as a 'proof by exhaustion' because you have exhausted all the situations that could occur.

Try it

The last activity illustrates some of the steps in a mathematical investigation.

First to get a feel for the problem, you tried some specific examples with numbers – this is known as **specialising**. From these examples, it looked like there might be a general result that if the first and last digits differ by 2 or more, the answer is always 1089. Suggesting a result like this which applies in general (in this case, to any three-digit number of this kind) is known as **generalising** and the statement is known as a **conjecture**. The final step is to **prove** that the conjecture is true for all possible numbers. In this example we proved the conjecture that if the first and last digit were the same, the final result was zero and the conjecture that if the first and last digit differed by one, the result was 198. There are many other ways of proving conjectures including using algebra which you will meet if you study maths further.

2.4 Using numbers: multiplication and division

In the previous section, we noted that the processes of addition and subtraction can be described in several different ways in questions and problems.

Activity 20 Words and notation

What words and notation describe the processes of multiplication and division? Can you suggest some practical situations when these operations are used?

Comment

The usual symbol for multiplication is ×. Examples of words that are used for multiplication are '5 **times** 6', '3 **lots** of 20', 'the **product** of 6 and 7' and on a cash withdrawal machine 'Please enter a **multiple** of £10'. Situations when you use multiplication are working out the cost of say eight books at £4.99 each.

The × symbol was first used by William Oughtred (1574–1660) in 1631.

published in German in

For division, a range of notations are used: 72 ÷ 8 or $\frac{72}{8}$ or 72/8 or $8\overline{)72}$ all mean 72 divided by 8. Practically, this is the same as **sharing** 72 items among eight people. We might also ask, 'How many times does 8 go into 72?'

For example, if six friends share the cost (£30) of a taxi home, each person would pay £30 ÷ 6 or £5 each.

You will probably do most multiplications and divisions on your calculator. However in order to check that your calculations are reasonable, it is useful to be familiar with multiplication and division by the digits 2 to 9, 10, 100. To investigate multiplying and dividing by 10, 100, 1000, and also to learn how to use the constant facility on your calculator, please work through, Section 2.1 of Chapter 2 of the Calculator booklet now.

Knowing the results of multiplying by the numbers from 2 to 9 can help you to work out calculations fairly quickly and confidently. But if you do not know these, it is possible to work out some calculations using the strategies we used earlier of breaking down calculations into steps that you can deal with.

Here are some techniques.

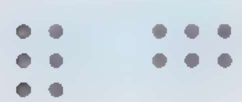
- To multiply by 5, you can multiply by 10 and then divide the answer by 2.
- To multiply by 4, you can double the number and then double it again.
- To multiply by 8, you can double the number three times.

You can adopt similar techniques for division as well.

- To divide by 4, you could halve the number and then halve it again.
- To divide by 5, you can divide by 10 and then multiply by 2.

Can you explain why these strategies for multiplication and division work? If you are not sure, try some examples out first to convince yourself that they do.

When you add two numbers together, the order does not matter: 2 + 4 is the same as 4 + 2; but what about multiplication and division? Is 3 × 2 the same as 2 × 3 and 4 ÷ 2 the same as 2 ÷ 4? How can you convince yourself that your answers are right?



When you multiply two numbers together it does not matter which order you do the multiplication in. So 3 × 2 is the same as 2 × 3. To convince yourself, imagine a diagram showing three rows of two dots. Turn this round so that it shows two rows of three dots. No dots have been added or lost, so the number of dots in both arrangements is the same and hence 3 × 2 = 2 × 3. This means that you can carry out the calculation in whichever order you find easier. Multiplication, like addition, is commutative.

However, the order you carry out division does matter: 4 ÷ 2 is not the same as 2 ÷ 4. Imagine you have four cakes to share between two people – this is not the same as sharing two cakes among four people!

Activity 21 Buying biscuits

Suppose a packet of biscuits costs £1.50. Use the methods above to work out, in your head, the cost of the following.

(a) 5 packets (b) 8 packets (c) 19 packets

Comment

There are many different ways of working out these calculations in your head. Here are a few approaches.

(a) Two packets would cost £3, so, doubling again, four packets would cost £6, and an extra packet would then make £7.50. (Alternatively, 10 packets would cost £15, so five packets would cost half this amount – £7.50.)

(b) Two packets would cost £3, so four packets would cost £6 and eight packets would cost £12.

(c) Ten packets cost £15, so 20 packets cost £30, and 19 packets will cost £1.50 less than this, so subtracting £1 gives £29 and then taking away the 50p gives £28.50.

Activity 22 Paper supplies

A college shop buys A4 pads of paper in bulk to sell to students at a cheap rate.

Each pack of paper contains 20 pads. If the shop wants 1500 pads for the term, how many packs should be ordered?

Comment

We need to find how many lots of 20 are in 1500, so the calculation is 1500 ÷ 20. If you imagine dividing some quantity into 20 heaps, one way to do it would be to divide it into 10 heaps and then divide each of those heaps into 2 heaps. So dividing a number by 20 is the same as dividing by 10 and then dividing by 2. (You can check this on your calculator. Is 1500 ÷ 10 ÷ 2 the same as 1500 ÷ 20?)

Dividing 1500 by 10 gives 150. To divide 150 by 2, you can either split the problem up by dividing both 100 and 50 by 2 and adding the results together to get 50 + 25 = 75 or write it out more formally like this:

$$\begin{array}{r}75\\ 2\overline{)150}\end{array}.$$

This is the same as saying 150 is 15 lots of 10.

Dividing this by 2 gives 7 (lots of 10) and 1 (lot of 10) left over. Then dividing the 'left over' lot of 10 by 2 gives 5.

There are other ways to work this out. For example, one pack contains 20 pads so five packs will contain 100 pads of paper. Hence 50 packs will contain 1000 pads and 25 packs will contain 500 pads. So 75 packs will be needed.

2.5 Using units of measurement

As you have seen from some of the problems earlier in this chapter, many numerical problems involve counting, in which the question is 'how many?' Everyday problems also often involve measurements, where you need to know 'how much?' there is of some quantity. Things are measured to find out how big, how long or how heavy they are. For these numbers to have any meaning, you need to specify what units the measurement is in. For example, if you are measuring the width of a window for some new glass, the glazier will want to know whether the measurement of 0.8 that you have given is in metres or yards or feet or some other unit.

There are hundreds of different units, but scientists and technologists of all nations have agreed to use a standard system of units. Everything you could ever want to measure can be measured using a few basic units, or combinations of them. These internationally agreed units, and the rules for their use, form the Système Internationale (International System), known as SI. This section explains the essentials of SI, which is sometimes referred to as the metric system. If you continue your studies, particularly in science or technology, you will encounter more examples of the use of SI units. However in some situations, you may also encounter some non-SI units, for example in the United Kingdom, miles are often used to measure distances.

To show you how the SI system works, we shall look at the units of length. The SI base unit for length is the metre, symbol 'm'. It is useful to know a rough idea of the size of a typical measurement in any given unit, so, for instance, the height of a typical domestic door is about 2 metres. If you are not familiar with the SI system, do try to get some practical experience of the size of the different units by noting where they are used in your life. For longer lengths, such as the distance between two towns, the SI system uses a multiple of the base unit, a kilometre (or km for short). The 'k' in 'km' and the 'kilo' in kilometre are examples of a **prefix** – a letter or word placed in front of another word. Prefixes show the scale of the measurement, and are standard for all SI units; 'kilo' means 'a thousand'. So 1 kilometre is the same as a thousand metres.

There are other prefixes for smaller units. So, the length of this page might be measured in centimetres (cm), where 'centi' means 'a hundredth'. A centimetre is one-hundredth of a metre so there are 100 centimetres in 1 metre. A UK 1p coin measures about 2 cm. For even smaller distances, we use millimetres (mm). The prefix 'milli' means 'a thousandth' so a millimetre is one-thousandth of a metre and there are 1000 millimetres in 1 m.

1 km = 1000 m

1 m = 100 cm

1 cm = $\frac{1}{100}$ m so 100 cm = 1 m

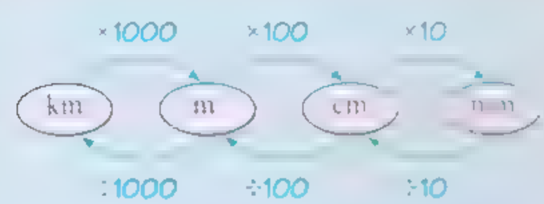
1 mm = $\frac{1}{1000}$ m so 1000 mm = 1 m

When you are using measurements in calculations, you often need to convert all the measurements into the same units before you start the calculation.

For example, suppose you wanted to convert 2.3 metres into centimetres. We know that there are 100 cm in 1 m. So to convert a measurement in m into cm, we need to multiply by 100.

So, 2.3 m = 2.3 × 100 cm = 230 cm.

You can see how to convert from one unit of length to another, in the diagram below. Notice that to convert from a **big** unit, say km, to a **smaller** unit, say m, you **multiply** by the number of metres in a kilometre. This makes sense – you will need a lot more of the smaller units. On the other hand, to convert a measurement in a **small** unit, say cm, to a **bigger** unit, say m, you **divide** by the number of centimetres in a metre.



Activity 23 Units of length

(a) Write 327 centimetres as metres.

(b) Write 6.78 metres as centimetres.

(c) Write 0.34 centimetres as millimetres.

Comment

(a) There are 100 cm in 1 m and we are converting from a small unit to a large unit, so we divide by 100 to change cm into m.

Hence, 327 cm = 327 ÷ 100 m = 3.27 m.

(b) There are 100 cm in 1 m. This time we are converting from a large unit to a smaller unit, so we multiply by 100.

Hence, 6.78 m = 6.78 × 100 cm = 678 cm.

(c) There are 10 mm in 1 cm. So 0.34 cm = 0.34 × 10 mm = 3.4 mm.

For some more practice, try Activity 24. Watch out for the unit conversions – always check that the final result makes sense. The most common mistake in this type of conversion is to muddle up multiplying and dividing, so you end up with a result that is a hundred or a thousand times too big or too small. Think about the actual measurements, and whether the final answer is reasonable, and you should avoid that trap.

Activity 24 Designing a kitchen

A kitchen measures 3.58 m by 2.45 m and the owner would like to put floor cupboards along one of the shorter sides. The cupboards from a DIY store are 400 mm wide. How many cupboards should the owner buy?

Comment

Here the measurements for the kitchen are given in metres and the floor cupboard units are measured in millimetres. To compare the two we need to have both measurements in the same units. It does not matter whether you work in metres or millimetres (or even centimetres) provided that you are consistent. I shall work in millimetres. To change metres into millimetres, multiply by 1000 as there are 1000 millimetres in a metre.

So 2.45 m = 2.45 × 1000 mm = 2450 mm.

We need to find out how many 400 mm cupboards will fit into this space. One way to do this is to divide the width of the wall, by the width of a cupboard.

The number of cupboards will be 2450 ÷ 400 = 6.125.

However you can't buy fractions of a cupboard. So the owner should buy six cupboards as seven will not fit. (Check that if you change the cupboard dimensions into metres by dividing by 1000, you get the same result.)

Notice that this is an example where practical rounding may not give the same result as mathematical rounding. If we had decided to fit cupboards along the long side of the kitchen (3.58 m), then using the same method as above, we would get 8.95 cupboards. Rounding this using the rules in Section 2.2 gives 9 as the rounded answer. But nine cupboards need 9 × 400 mm = 3600 mm or 3.6 m. So nine cupboards won't fit! So here we must round down to 8, and there's enough space left over for a bottle rack as well as the trays.

So far, in this section, we have just considered measurements of length. Of course, there are other quantities like mass which may be measured in grams (g) and kilograms (kg). For example, sugar is often packed in 1 kg bags. Sometimes you may hear people talking about their 'weight' in kilograms. This is where the everyday use of a word differs from the technical or scientific definition. There is a subtle difference between mass and weight – your mass is the amount of matter in you but your weight depends on

the gravitational pull on you as well. When you see a measurement called 'weight' and quoted in grams, kilograms or similar units, you can use this in your calculations although strictly it should be called 'mass'.

The advantage of the metric system is that the same prefixes apply, whatever the base unit, so 'm' for 'milli' still means 'one-thousandth', 'k' for 'kilo' means 'one thousand', and so on. For smaller quantities, such as the amount of vitamin in a vitamin tablet, the microgram is used, symbol µg. 'µ' is the prefix for 'micro' which means 'one-millionth'. 'µ' is actually a letter of the Greek alphabet, called 'mu', although it is usually read as 'micro' when used with units.

1 tonne = 1.102 311 31 tons

For larger masses, such as the mass of a car, the metric tonne (t) is used, where one tonne is 1000 kilograms. Note the spelling – if you see 'tonne' written by itself, that means metric tonne. If you see 'ton', that implies the older, non-metric ton. Strictly speaking, the tonne is metric but not SI, although it is used alongside SI units for convenience.

The following diagram shows the units for mass that you'll need in this course, and Activity 25 gives you a chance to practise some unit conversions.

Activity 25 Vitamin tablets

A multivitamin tablet contains various vitamins and minerals. The table below shows the recommended daily amounts (RDA) for each of these, as shown on the packet.

Vitamin tablets	
Vitamin B12	1 µg
Vitamin C	60 mg
Vitamin E	10 mg
Folic acid	200 µg

From this list, can you tell which RDA is the largest amount?

Comment

To compare these different amounts, it helps to have each quantity measured in the same units. Micrograms are smaller than milligrams, so convert everything to micrograms.

Since 1 mg = 1000 µg, to convert mg into µg, multiply by 1000.

The RDA for Vitamin C is 60 mg = 60 × 1000 µg = 60 000 µg.

The RDA for Vitamin E is 10 mg = 10 × 1000 µg = 10 000 µg.

Since the RDA for Vitamin B12 is 1 µg and the RDA for folic acid is 200 µg, the amounts in order, largest first, are Vitamin C, Vitamin E, folic acid and Vitamin B12. You can see that you need the most Vitamin C per day, compared with the other nutrients.

The last quantity that will be covered here is volume. In everyday use, the litre (l) is often used and you may have seen 1 litre cartons of fruit juice. Just as for tonnes, this is a metric unit that is used alongside SI for convenience. By now, you'll probably have guessed the pattern of units. Working from litres, the two commonest multiples are the centilitre (cl), which is one-hundredth of a litre (just as 1 centimetre is one-hundredth of a metre), and the millilitre (ml), which is one thousandth of a litre. Try Activity 26 to see how these prefixes work.

Activity 26 A dose of medicine

A new bottle of medicine contains 14 cl. The dose is 5 ml. How many doses can be given from this bottle?

How would you describe, perhaps to a friend, how to convert between millilitres, centilitres and litres? Use a diagram if it helps.

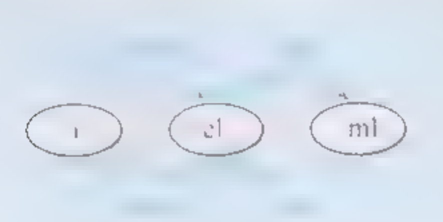
Comment

To find the number of doses, the volume of the bottle and the dose need to be in the same units. There are 10 ml in 1 cl. So 14 cl = 14 × 10 ml = 140 ml.

So the number of doses in the bottle is 140 ÷ 5 or 28 doses.

There are 10 ml in 1 cl and a 100 cl in 1 litre. So to convert litres into cl **multiply** by 100. To convert cl into ml **multiply** by 10. To convert ml into cl **divide** by 10 and to convert cl into l **divide** by 100.

This is summarised in the following diagram.



The other common measurement is time. This is where the SI unit – the second – is used in science and technology, but in everyday life we usually use non-SI units – days, weeks, years. You may come across milliseconds, although these are not often used outside the science laboratory. As you might expect, one millisecond is one-thousandth of a second.

This section has dealt with units of measure. These have included large measures like kilometres and small ones like millimetres. When any measurement is taken, there is always an element of rounding, depending on the instrument you are using – for example, you might round to the nearest millimetre or centimetre. The next section looks again at rounding numbers.

2.6 More rounding

You have seen that there is a limit to how precisely quantities can be measured, so this section looks at two more types of rounding that are frequently used: rounding to decimal places and rounding to significant figures.

Rounding to decimal places

Suppose you have two pictures that you want to put up on a wall so that the picture hooks divide the length of the wall into thirds. If the wall measures 3.47 metres, where should the hooks be positioned?

The two hooks will split the wall into three equal sections, so to find the horizontal measurement of each hook from the edge of the wall, you need to calculate $3.47 \div 3$.

An approximate answer is 1 m, but this is not accurate enough for this purpose. Using a calculator gives the answer 1.156 666 667 m. As measurements are only likely to be made to the nearest centimetre or millimetre, this answer needs to be rounded.

Decimal fractions are often rounded to a given number of **decimal places** (d. pl. or d.p.). The decimal places are to the right of the decimal point.

1.156...

first d.p. second d.p. third d.p.

In this example, rounding to the nearest centimetre is the same as rounding to two decimal places, because the measurement is in metres. You can use the '5 or more' rule here in a similar way to the examples in Section 2.2. To round to two decimal places, find the digit in the second decimal place. If it helps, draw a line after this digit to highlight it. Then look at the next digit on the right, in the third decimal place, to see if it is '5 or more'. In this case, the digit is 6, so the number is rounded up to 1.16.

is this digit 5 or more?

1.15|6666667

second d.p.

Similarly, if 1.156 666 667 is rounded to three decimal places, the answer is 1.157.

When you are rounding a number, how do you decide how many decimal places are appropriate? Since the original wall measurement was given to the nearest centimetre, it makes sense to round the distance from the wall to the hook to the nearest centimetre as well. If you do round an answer, remember to quote the rounding you have used, after the number. For example, you could write 'The distance from the wall to the hook is 1.16 m (to 2 d.p.)'.

Activity 27 Rounding to decimal places

(a) Round the following numbers as indicated.

(i) 3.425 (1 d.p.)

(ii) 24.0068 (2 d.p.)

(iii) 0.936 52 (3 d.p.)

(b) Can you think of any situations where you might need to round a number to a particular number of decimal places?

Comment

this digit is less than 5 so round down

3.4|25

first d.p.

(a) (i) The digit to the right of the first decimal place is 2, which is less than 5, so 3.425 rounds down to 3.4 (to 1 d.p.).

(ii) The digit to the right of the second decimal place is 6, which is more than 5, so 24.0068 rounds up to 24.01 (to 2 d.p.).

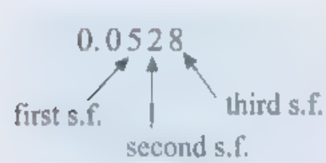
(iii) The digit to the right of the third decimal place is 5, which is 5 or more, so 0.936 52 rounds up to 0.937 (to 3 d.p.).

(b) Money calculations in pounds and pence often involve rounding to two decimal places, as in £1.27 for example.

To get more of a feel for calculations involving rounding, now work through Section 2.2 of Chapter 2 of the Calculator booklet.

Rounding to significant figures

An alternative way of rounding numbers is to use **significant figures** (or s.f. or sig. fig.). In a number, the first significant figure is the furthest **non-zero** digit to the left and it gives you an indication of the size of the number. For example in 302, the first significant figure is 3 and it tells you that the number is between 300 and 400. The second significant figure is zero and the third significant figure is 2.



In 0.0528 the first significant figure is 5 and it tells you that this number is between 5 hundredths and 6 hundredths. The second significant figure is 2 and the third significant figure is 8.

To round a number to, say, two significant figures, first identify the first two significant figures. Then look at the next digit on the right. If it is 5 or more round up; 4 or less, round down.

In 0.0528, the first two significant figures are the 5 and 2. The next digit on the right is 8. As this is more than 5, the number rounds up to 0.053 (to 2 s.f.). Similarly to 1 s.f., the number is 0.05.

Significant figures are often used when working out an estimate for the answer to a calculation. Rounding the numbers in the calculation to one (or possibly two) significant figures makes the estimate easy to work out as you will see in Activity 28.

Activity 28 Rounding to significant figures

(a) Round the following numbers as indicated.

(i) 16 840 (3 s.f.)

(ii) 0.005 82 (1 s.f.)

(iii) 0.0195 (2 s.f.)

(iv) 3078 (2 s.f.)

(b) Four items are to be sent in a parcel by post. The items weigh 1.91 kg, 4.78 kg, 3.25 kg and 6.93 kg. By rounding each weight to 1 s.f., make an estimate for the total weight. Then work out the total weight using your calculator.

Comment

(a) (i) The first three significant figures are 1, 6 and 8. The next digit is 4 and, as this is less than 5, 16 840 rounds down to 16 800 (to 3 s.f.).

(ii) The first non-zero digit on the left is the 5, so this is the first significant figure. The next digit is 8 and, as this is more than 5, the number 0.005 82 rounds up to 0.006 (to 1 s.f.).

(iii) The first two significant figures are 1 and 9. The next digit is 5, so 0.0195 rounds up to 0.020 (to 2 s.f.). Although 0.020 has the same value as 0.02, the zero in the third decimal place is the second significant figure and so is included. For a measurement the number of significant figures indicates the level of precision. The range of measurements that round to 0.020 is less than the range of measurements that round to 0.02. Measurements from 0.0195 up to but not including 0.0205 round to 0.020. Measurements from 0.015 up to but not including 0.025 round to 0.02.

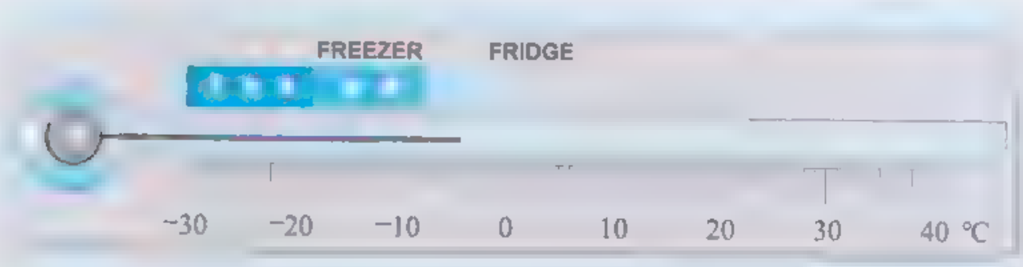
(iv) The first two significant figures are 3 and 0. The next figure is 7, so the number rounds up to 3100 (to 2 s.f.).

(b) Rounding to 1 s.f., the weights are 2 kg, 5 kg, 3 kg and 7 kg, giving a total of 17 kg. The calculator answer is 16.87 kg which is similar to the estimate. So the total weight of the items is 16.87 kg.

2.7 Negative numbers

It took mathematicians a long time to get to grips with negative numbers, so do not be surprised if you need to work slowly and carefully through this section.

Negative numbers are used when you want to describe a quantity that is less than zero. For example, a label on frozen food often suggests that the temperature at which the item is stored should be '$^{-}18$ °C or lower'. This number is read as 'minus 18 degrees Celsius' and it means 18 degrees below zero. Keeping food at the correct temperature is important to prevent bacteria growing and spoiling the food. In this course, negative numbers are denoted by using a raised negative ($^{-}$) sign to distinguish it from the subtraction sign (–). (Elsewhere, this distinction is not always made and the subtraction symbol is used for both negative numbers and the subtraction operator.)



You can check the temperature of a fridge or freezer using a special thermometer which shows the temperature between about $^{-}40$ °C and 40 °C. On the thermometer above each of the small divisions represents one degree C. This means that the temperature recorded is $^{-}4$ °C. You can see on the thermometer that the temperatures increase to the right and a temperature of $^{-}10$ °C is warmer than a temperature of $^{-}20$ °C. The packaging on frozen

food indicates how long the food can be stored at different temperatures by using a star rating. On the thermometer above, one star is the same as a temperature range from −8 °C to 0 °C.

Activity 29 Freezer stars

On the thermometer above, what is the temperature range that corresponds to two stars and three stars?

Comment

The two stars symbol corresponds to a temperature range from −18 °C to −8 °C and three stars from −30 °C to −18 °C.

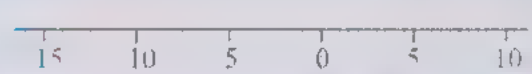
Addition and subtraction

Negative numbers are used in many other practical applications too. For example, banks and other companies use negative numbers to represent overdrawn accounts or a debt. For example, if you had £50 in your account but then withdrew £60, your account would be overdrawn by £10 and this might be recorded as −£10 or £10D (debit). You may also have seen negative numbers on maps where they are used to record the depth below sea level or on video recording equipment where negative numbers are sometimes used to indicate position.

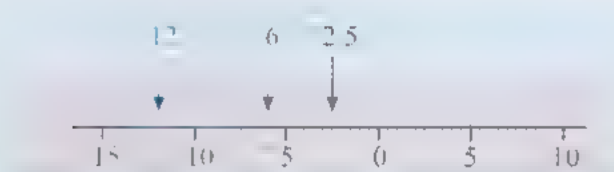
Negative numbers can be shown on the number line in a similar way to the temperatures on a thermometer. For example, the number line below shows the numbers −0.5, −1.8 and −2.4. These numbers may also be read as 'negative nought point five, negative one point eight and negative two point four'. Notice that the numbers increase to the right. So −1 is greater than −3.

Activity 30 Negative numbers

Mark the numbers −2.5, −6 and −12, on the line below.



Comment

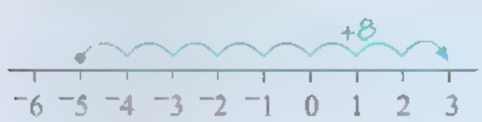


The number line can also be used to show how calculations can be carried out with negative numbers. First check how this works with positive numbers by looking at the sum 2 + 4.

Start at 2 and move 4 units to the right to get the answer 6.

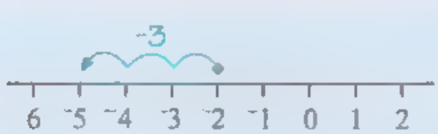
The same method works if the starting number is negative. For example suppose you are overdrawn by £5, so your account balance is recorded as $^{-}5$. If you then pay £8 into your account, £5 will pay off your debt and £3 will be recorded as your new balance, so $^{-}5 + 8 = 3$.

On the number line the starting number is $^{-}5$, moving 8 units to the right gives the answer as 3.



Now consider subtraction, for example 6 – 4. Here the starting number is 6, but this time move 4 units **to the left** to get to the answer 2.

Similarly if you are overdrawn by £2 and then withdraw a further £3, you will then be overdrawn by £5. This is represented on the number line by starting at $^{-}2$ and moving 3 units to the left to get the answer $^{-}5$. This is written $^{-}2 - 3 = {}^{-}5$.



Activity 31 Adding and subtracting positive numbers

Work out the following calculations, using a number line if you like.

(a) 3 – 7

(b) 2 + 5

(c) $^-$10 – 6

(d) 4 + 2

Comment

(a) Start at 3, move 7 units to the left to get 3 – 7 = 4

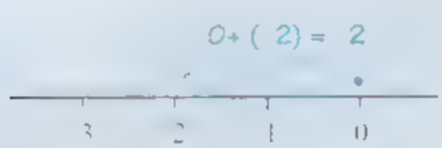
(b) Start at 2, move 5 units to the right to get 2 + 5 = 3

(c) Start at $^-$10, move 6 units to the left to get $^-$10 – 6 = 16

(d) Start at 4, move 2 units to the right to get 4 + 2 = 2

The calculations in Activity 31 involved adding and subtracting positive numbers. The process is similar when adding and subtracting negative numbers.

Let's consider a simple case first: 0 + ($^-$2). If you add a number to zero you end up with the same number. For example, 0 + 6 = 6; 0 + 23 = 23, and so on. So 0 + ($^-$2) = (2). On the number line, starting at 0 and ending at 2, involves moving 2 units to the left, which is the same operation as subtracting 2.



> Adding a negative number is the same as subtracting the corresponding positive number.

So for example:

30 + (20) = 30 – 20 = 10 and 30 + (20) = 30 – 20 = $^-$50.

As a practical example, suppose you open a new account and then transfer in a debt of £20 from another account. Your new account will then show a balance of 20, which is just the same as opening an account and then withdrawing £20 immediately.

Now what happens when you subtract a negative number? This is quite difficult to visualise, so try thinking of subtraction in a different way. For example, for the calculation 30 – 20, instead of saying '30 take away 20 is …', you can say 'what do I have to add on to 20 to get 30?'.

In the same way, $0 - (^-2)$ can be interpreted as what do I have to add on to $(^-2)$ to get 0? From the number line, to get from $^-2$ to 0, you have to move 2 units to the right which is the same as adding on 2. This is also true more generally.

> Subtracting a negative number is the same as adding the corresponding positive number.

For example:

$30 - (\ 20) = 30 + 20 = 50$ and $\ 30 - (\ 20) = \ 30 + 20 = \ 10$.

Work out the following calculations.

(a) $20 + (\ 5)$

(b) $^-20 + (\ 5)$

(c) $20 - (\ 5)$

(d) $^-20 - (\ 5)$

Comment

(a) $20 + (\ 5) = 20 - 5 = 15$

(b) $\ 20 + (\ 5) = \ 20 - 5 = \ 25$

(c) $20 - (\ 5) = 20 + 5 = 25$

(d) $^-20 - (^-5) = {}^-20 + 5 = {}^-15$

Work through Sections 2.3 and 2.4 of Chapter 2 of the Calculator booklet now.

Multiplication and division

In the calculator section, you found that $(^-2) \times 4 = {}^-8$ or alternatively $4 \times (^-2) = \ 8$. This is the same as four lots of $(^-2)$ or $(^-2) + (^-2) + (^-2) + (^-2)$. It can be written as $\ 2 - 2 - 2 - 2$. So starting at $\ 2$ and then carrying out the subtractions does give an answer of $^-8$. You also found that $(^-4) \times (^-2) = 8$. The product of any two negative numbers can be shown to be positive.

Now consider $(\ 6) \div 2$. An alternative way of considering this problem is to say: 'What do I have to multiply 2 by, to get $(^-6)$?' Since 2 is positive, and $\ 6$ is a negative number, the answer must be negative, so $2 \times (^-3) = {}^-6$ and hence $(^-6) \div 2 = \ 3$. Similarly for $(\ 6) \div (^-2)$, since you would need to multiply $(^-2)$ by 3 to get $(\ 6)$, $(\ 6) \div (^-2\ ) = 3$. The statements below summarise these results.

> Multiplying (or dividing) a negative number by a positive number (or vice versa) gives a negative number.
>
> Multiplying (or dividing) a negative number by a negative number gives a positive number.

Alternatively, if the two numbers being multiplied together (or divided) have different signs, the result is negative and if they have the same sign, the result is positive.

Activity 33 Multiplying and dividing negative numbers

Work out the following calculations.

(a) $(^-3) \times 5$

(b) $(^-3) \times (^-5)$

(c) $(^-10) \div 5$

(d) $(^-10) \div (^-2)$

Check your results on your calculator.

Comment

(a) $(^-3) \times 5 = {}^-15$, since a negative number multiplied by a positive number gives a negative result.

(b) $(^-3) \times (^-5) = 15$, since a negative number multiplied by a negative number gives a positive result.

(c) $(^-10) \div 5 = {}^-2$, since a negative number divided by a positive number gives a negative result.

(d) $(^-10) \div (^-2) = 5$, since a negative number divided by a negative number gives a positive result.

2.8 Putting it all together

In the examples you have worked through so far, all the information you needed was given to you and sometimes you were given a hint on how to tackle the problem too. Using maths in your own life isn't like that. You will need to decide how to tackle the problem and also what information you need and that can be a lot more challenging than working through some examples in a textbook. Real life maths can be messy and some of the decisions you make may depend on your priorities, such as cost or time. For example on a DIY store's website the following instructions were given for calculating how many rolls of wallpaper were needed to decorate a room.

> Calculate the number of rolls you will need, using the method outlined below.
>
> Step 1: A standard roll of wallpaper is approximately 10.5 m (33 ft) long and 530 mm (1 ft 9 in) wide. If you measure the height of the walls from skirting to ceiling, you can work out how many strips of paper you can cut from a standard roll – four strips is about average.
>
> Step 2: Measure around the room (ignoring doors and windows) to work out how many roll widths you need to cover the walls. Divide this figure by the number of strips you can cut from one roll to calculate how many rolls you need to buy. Make a small allowance for wastage.
>
> (B&Q, 2005)

Activity [illegible] Following the instructions

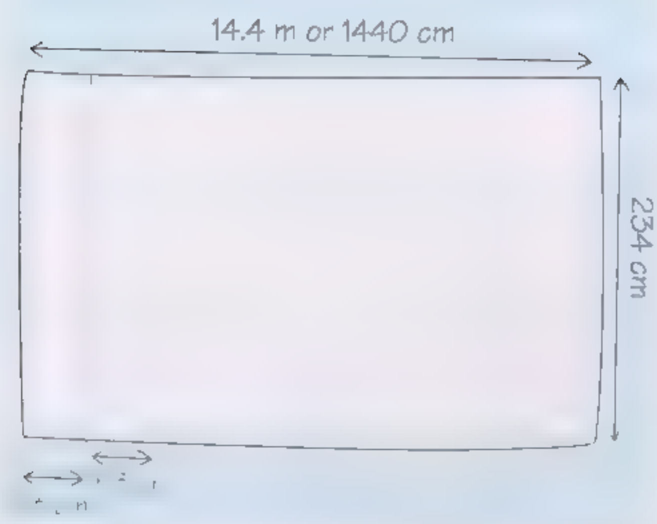
(a) Read through the extract above. Highlight the important information. What did you do to make sense of these instructions?

(b) Using your own measurements for a room (or assume the room measures 3.2 m by 4 m and the height of the walls is 2.34 m), work out how many rolls will be needed.

Comment

(a) This is how Tony, a student, tackled the problem. His notes are given below.

> I read all the instructions through first of all to get an overall idea of what I would need to do. Then I went back and highlighted the key bits of information – the dimensions of the roll of wallpaper and the instructions for what I needed to measure. I wasn't sure what 'ignoring doors and windows' meant in the instructions for measuring round the room. Does it mean you shouldn't measure the part of the wall occupied by the door and window, or that you should measure the width and length of the room as though there wasn't a door and window there? I discussed this with a friend and opted for the latter, since that would overestimate the amount required and I didn't want to end up with too few rolls of wallpaper.
>
> Then I concentrated on the calculations. There are three calculations – working out how many strips of paper you can cut from a standard roll; working out how many roll widths there are round the room and then calculating how many rolls are needed. For the second calculation, I considered a simpler problem first, say a wall that was 10 m long and a roll of paper that was 0.5 m wide – that helped me to see that I needed to divide the length of the wall by the width of the roll since the question was 'how many widths are there along the wall?'. I used the measurements to sketch out a diagram so that I could visualise the problem more easily as well. My room measured 3.2 m by 4 m, and the skirting to ceiling height was 2.34 m, so the distance round the room was 3.2 m + 4 m + 3.2 m + 4 m = 14.4 m. So the description ended up looking like this (I added my calculations later, after checking how to convert metres to centimetres):
>
> Step 1: A standard roll of wallpaper is approximately 10.5 m (33 ft) long and 530 mm (1 ft 9 in) wide. If you measure the height of the walls from skirting to ceiling, you can work out how many strips of paper you can cut from a standard roll – four strips is about average.
>
> Step 2: Measure around the room (ignoring doors and windows) to work out how many roll widths you need to cover the walls. Divide this figure by the number of strips you can cut from one roll to calculate how many rolls you need to buy. Make a small allowance for wastage.



(b) The first calculation is to work out how many lots of the height (2.34 m) you can get from a roll of wallpaper that is 10.5 m long. Four strips will use 4×2.34 m = 9.36 m and 5 strips will use 5×2.34 m = 11.7 m. So you will be able to get four but not five strips from each roll.

Next you need to work out how many strips there are around the room. If you already have a roll of wallpaper, you could count how many strips you'll need by measuring along the wall with the roll of paper. If you don't have any wallpaper, you can calculate how many 53 cm wide strips will fit in to 14.4 m.

14.4 m = 14.4×100 cm = 1440 cm.

So the number of strips is $1440 \div 53 \approx 27.2$.

Rounding this up gives 28 strips altogether. As there are four strips to the roll, $28 \div 4 = 7$ rolls will be needed. This will allow some wastage because paper is not put over the doors or windows. However if the paper has a large pattern that needs matching up it might be worth buying an extra roll to be on the safe side.

Reading maths

Reading a piece of mathematics (or a mathematical extract as above) does require a more detailed and active approach than some other types of reading, so it is worth taking a few moments to think about the strategies that Tony used and also ones that you have used while working through this chapter. Some useful tactics are summarised in the box below, but you might want to add your own ideas to the list as well.

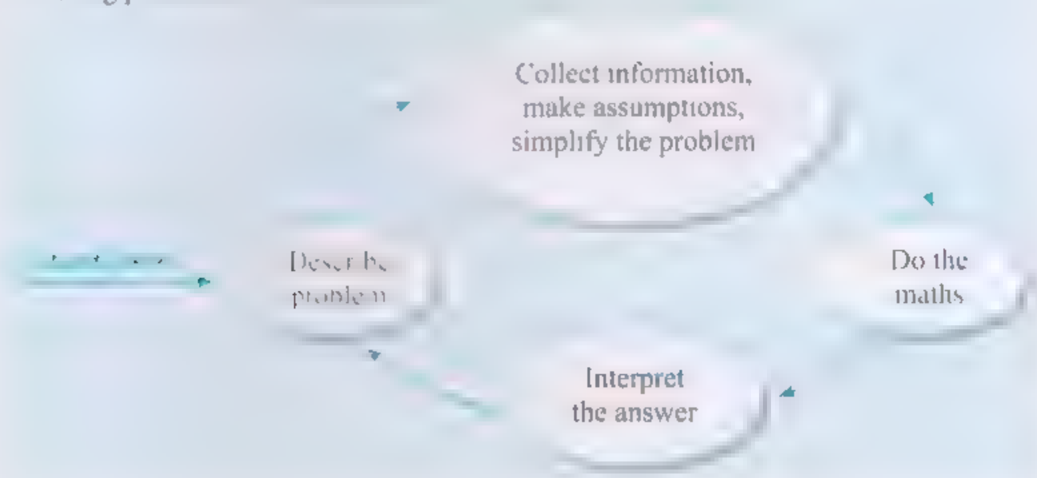
Reading mathematics

- Read carefully and check you understand any special terminology, symbols or abbreviations, and also what you have to do.
- Highlight (or underline) key pieces of information.
- Check that you have all the information you need to hand, including skills and techniques learned earlier.
- Add extra lines of working if that helps.
- Draw a diagram (to help you visualise the problem) and put the information you have on it.
- Mark parts that you find difficult. You may want to come back to these or talk through the ideas with a friend or your tutor.

The mathematical modelling cycle

Activity 34 also illustrated some of the main steps in solving a real problem mathematically. These steps can be summarised in the 'mathematical modelling' cycle shown below.

Solving problems with mathematics



There are four main steps in this cycle.

- Describing the problem concisely so that you are clear what you are trying to work out. This may involve discussing the problem with others.
- Collecting information and making assumptions so that you can use mathematics to solve it.
- Deciding what mathematical techniques to use, and carrying out these calculations.
- Working out what your results mean practically and checking that they are reasonable. If they are not, you may need to refine the assumptions and go round the cycle again.

The first step with the wallpaper problem was to describe the problem carefully – here it was 'how many rolls of wallpaper are needed?', Then Tony needed to check what information he had (the dimensions of the roll of wallpaper) and what extra information needed to be collected – the height of the walls and the distance round the room. He made some assumptions here by ignoring the sections with the door and the windows in. He also ignored any pattern on the wallpaper that would need matching up. These assumptions made the problem easier to tackle and the maths easier too.

Then he worked out the calculations he needed to do, checking that the measurements were all in the same units of centimetres. The final step was to interpret the answer, rounding it appropriately and to check that the answer of 7 rolls seemed reasonable. For example, an answer of 700 rolls would have suggested a mistake somewhere in the calculations and asking for 6.792 452 83 rolls in the shop would not have been helpful either!

At this stage it is also worth having another look at the assumptions that were made to see if the calculations need refining in any way. For example, 'How confident are you that there will be enough wallpaper to match any pattern? Should you get another roll or just calculate the amount more accurately?' That last question may depend on the cost of a roll of wallpaper that you have chosen. If it is very expensive, a more detailed calculation might be better than just buying an extra roll. Then you would go round the modelling cycle again!

2.9 Taking stock

Now that you have reached the end of this chapter, it would be a good time to spend a few moments thinking back over your studying so far and considering any changes you would like to make for the rest of your studies.

Activity 35 How are you getting on?

Use the questions below as a guide to try to identify when you study well and what areas of your studying you would like to improve. Write down any changes that you would like to make before studying the next chapter.

- Did you find long or short study sessions more productive?
- Did the time of day make a difference?
- Were there times when you found concentrating difficult, and, if so, can you think why? What might help?
- Can you identify good times to work and what kinds of activities to do at which times?
- Were there particular sections that you found difficult? How did you deal with them?

Now, look back over your notes.

- Do your notes summarise all the main points?
- Can you still understand them?
- Do you need to include any more details in your notes, or would they be clearer if you left some details out?
- Would one of the other forms of note taking from Chapter 1 work better for you?

Finally, consider what mathematics you have learned from this chapter.

- Where can you use and practise the skills you have learned, e.g. the mental strategies or the rounding?
- Are there any topics that you need help with or need to look at again?

This process of spending a few minutes thinking about what you have achieved and what difficulties you have encountered is well worth following at the end of each chapter. It might suggest alternative study strategies that you could try for the next chapter.

Study checklist

You should now be able to:

- work out everyday problems using addition, subtraction, multiplication and division
- appreciate that strategies can be used to work out calculations on paper or in your head
- understand the SI system of measurement for length, mass and volume
- use rounding and estimation and check results from your calculator
- use negative numbers
- use some strategies for reading a piece of mathematics
- appreciate the steps in solving a problem mathematically.

3 Same and different

The main theme of this chapter is looking at how numbers can be expressed in different ways, and how these can make calculations or problems easier to tackle. By the end of the chapter, we hope that you will feel more confident in dealing with fractions, decimals and percentages in everyday life. These techniques open up a whole range of situations, where maths can help you to make decisions or solve problems. At some point on the course, we do expect you either to get stuck or to find the mathematics rather challenging, so this chapter also introduces some of the basic strategies you can use when faced with a new problem. We hope these ideas will help you as you work through the rest of the course. As well as writing numbers in an appropriate form, we shall also be considering how to write good clear mathematics, so that other people (including your tutor and you in a few weeks'!) can follow your ideas and mathematical solutions easily.

Before you start work on the activities in this chapter, skim through it to see what it contains. Make a note of the subheadings and activity headings that you recognise and those that appear new to you. Then based on your experience of the previous two chapters, try and estimate how long this chapter is going to take you. Make a note of your estimate in the margin. The time you take will depend on your previous experience and your mathematical knowledge, everyone is different. However this chapter does contain some essential ideas that will be needed later in the course, so it is important to work through the activities thoroughly on the topics that are new to you or perhaps long forgotten!

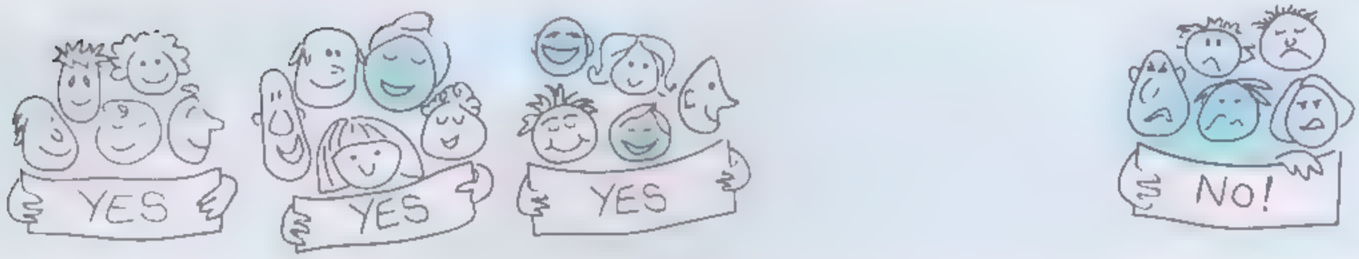
3.1 Fractions

Most people use fractions in their everyday life when they talk about time (half-past ten), parts of pizzas and cakes (halves and quarters) or when shopping (two-thirds off marked prices). You may also see fractions in headlines. The following headline appeared in a press release from the British Wind Energy Association in June 2005.

> Three-quarters of people in Wales believe wind farms are necessary says new poll.
>
> (BWEA, 2005)

What did you think when you read the headline – perhaps that this view was quite strongly supported? Does it tell you how many Welsh people actually think in this way?

Well, no it doesn't – it just tells you the **proportion** of the group of people who were interviewed, who thought in this way. If you got all the people who were interviewed together, you could arrange them into four groups with the same number of people in each group and then the people in three of the groups would have supported this view.



So if there were only four people interviewed, three would have agreed. If there were 4000 people interviewed, 3000 would have agreed, and so on. So how much notice you should take of the headline would probably depend on both the number of people who were interviewed and how they were selected. Interviewing a lot of people who had been selected at random may give a better indication of people's views than interviewing just a few people who lived a long way from any wind farm. Fortunately, the information on who was interviewed was provided in the press release.

In the survey, 500 Welsh people were interviewed. By dividing this group into quarters, work out how many of the people agreed with the view in the headline.

Comment

Three-quarters of the 500 people agreed.

First, split the group into quarters by dividing by 4: $500 \div 4 = 125$.

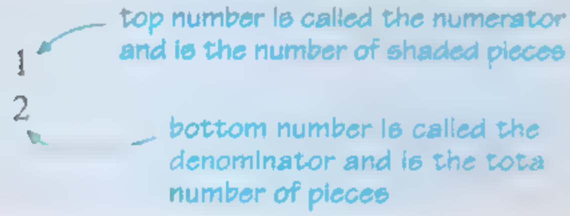
So three-quarters of the group will be three lots of 125 people.

Since $3 \times 125 = 375$, there were 375 people who agreed with the statement.

In a similar way, you can make sense of most everyday fractions by thinking of dividing the amount or number into equal parts and then considering how many of these parts you need. However there will be occasions, particularly if you go on to study more mathematics, science or technology courses, when you will need to carry out more involved calculations involving fractions. The next part of this section introduces fractions both practically, through folding paper, and in a more formal way and then considers some calculations that you can work out using your calculator.

Take a piece of blank A4 paper and fold it in half, creasing along the fold. Open it up and shade the left-hand side.

Since the paper has been divided into two identical pieces, each piece is half of the original. This fraction is written as $\frac{1}{2}$:



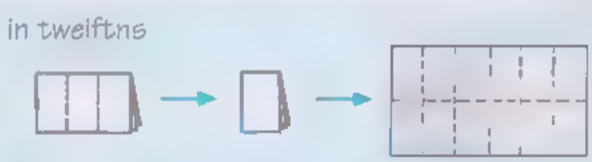
and read as 'one-half'.

[illegible]

Now fold the paper back along the crease and then in half again along the long side. If you open up the paper, you should see four pieces of the same size with two of them shaded. So the paper is now divided into quarters and the fraction of the paper shaded is $\frac{2}{4}$.

How many times do you think you can carry on folding a piece of paper in half? Try it! If the paper is 0.1 mm thick, try to estimate the height of a piece of paper that has been folded eight times.

Since you haven't altered the shading in any way, this shows that one half is the same as two quarters or $\frac{1}{2} = \frac{2}{4}$. Now fold the paper back into quarters along the crease lines and then fold into three equal pieces or thirds along the long side. If you now open the paper up, you can see that there are 12 equal pieces. These pieces are 'twelfths', of which six are shaded, so $\frac{6}{12}$ of the paper is shaded. This fraction also represents the same amount as $\frac{1}{2}$.



You can carry on folding the paper into smaller and smaller pieces and each time you open up the paper, it will be divided into smaller fractions, but half of it will still be shaded. The fractions which represent the shaded part are all equivalent to each other.

When you folded the original piece of paper in half the second time, you doubled the total number of pieces on the paper (to four) and also doubled the number of shaded pieces (to two). This is the same as multiplying both the top and the bottom of the fraction by two.

If you multiply the top and the bottom of any fraction by the same number (not zero!), you will get a fraction that is equivalent to the original one.

$$\frac{1}{2} \underset{\times 2}{\overset{\times 2}{=}} \frac{2}{4} \underset{\times 3}{\overset{\times 3}{=}} \frac{6}{12} \underset{\times 10}{\overset{\times 10}{=}} \frac{60}{120}$$

You can also generate equivalent fractions by dividing the top and the bottom of the fraction by the same number (not zero).

$$\frac{40}{60} \underset{\div 10}{\overset{\div 10}{=}} \frac{4}{6} \underset{\div 2}{\overset{\div 2}{=}} \frac{2}{3}$$

This process is known as 'cancelling'. If there is no whole number that can be divided into both the top and the bottom, the fraction is said to be in its **lowest form**. Answers are usually left in this form, as they are easier to visualise and understand.

Activity 37 Equivalent fractions

(a) Fill in the missing numbers in the following.

(i) $\frac{2}{3} = \frac{?}{18}$ (ii) $\frac{4}{5} = \frac{?}{20}$

(b) Which of the following fractions are equivalent to each other?

(i) $\frac{7}{8}$ (ii) $\frac{49}{64}$ (iii) $\frac{35}{40}$ (iv) $\frac{8}{9}$

(c) Cancel the following fractions into their lowest terms.

(i) [illegible] (ii) [illegible] (iii) [illegible] (iv) $\frac{60}{12}$

Comment

(a) (i) What do you have to multiply the denominator of 3 by to get 18? The answer is six, so this is the number that is used to multiply the top and the bottom. The missing number is 12.

$$\frac{2}{3} \underset{\times 6}{\overset{\times 6}{=}} \frac{12}{18}$$

(ii) Since $5 \times 4 = 20$, multiply the top and the bottom by four. The missing number is 16.

$$\frac{4}{5} \xrightarrow{\times 4} \frac{16}{20}$$

(b) Multiplying the top and the bottom of the first fraction by five gives

$$\frac{7}{8} \xrightarrow{\times 5} \frac{35}{40}$$

Only these two fractions are equivalent.

(c) (i) Dividing the top and the bottom by 2 gives

$$\frac{\cancel{2}^{1}}{\cancel{8}_{4}} = \frac{1}{4}.$$

(ii) Dividing the top and the bottom by 10 (or by 5 and then by 2) gives

$$\frac{\cancel{10}^{1}}{\cancel{30}_{3}} = \frac{1}{3}.$$

(iii) Dividing the top and the bottom by 9 and then by 3, gives

$$\frac{\cancel{27}^{3}}{\cancel{81}_{9}} = \frac{\cancel{3}^{1}}{\cancel{9}_{3}} = \frac{1}{3}.$$

(iv) Dividing the top and the bottom by 4 and then 3, gives:

$$\frac{\cancel{60}^{15}}{\cancel{12}_{3}} = \frac{\cancel{15}^{5}}{\cancel{3}_{1}} = \frac{5}{1} = 5.$$

(Of course it does not matter which number you divide by first.)

Cancelling is useful when you are trying to write one quantity as a fraction of another. For example, suppose you carried out your own survey and found that 240 people out of a group of 300 thought that junk food should not be available for school dinners. The fraction of the group with this view is $\frac{240}{300}$.

This can be cancelled down by dividing the top and bottom by 10, and then by 6. So $\frac{240}{300}$ is equivalent to $\frac{4}{5}$.

$$\frac{240}{300} \underset{\div 10}{\overset{\div 10}{=}} \frac{24}{30} \underset{\div 6}{\overset{\div 6}{=}} \frac{4}{5}$$

If you find it difficult to spot the numbers to divide by, try to work systematically by trying 2, 3, 5, ... in turn. You can check these results by using your calculator as you will see shortly.

Activity 3[illegible] Fractions of a group

From a survey of a group of 560 people, 245 say that they have studied a college course during the last year and 140 say that they have studied an OU course.

(a) What fraction of the total group has studied a college course?

(b) What fraction of the total group has studied an OU course?

Comment

(a) The fraction who have studied a college course is $\frac{245}{560}$. This can be cancelled by dividing the top and bottom by 5 and then by 7:

$$\frac{\cancel{245}^{49}}{\cancel{560}_{112}} = \frac{\cancel{49}^{7}}{\cancel{112}_{16}} = \frac{7}{16}.$$

So the fraction who have studied a college course is $\frac{7}{16}$

(b) The fraction who have studied an OU course is $\frac{140}{560}$.

Cancel by dividing the top and bottom by 10 then by 7 and finally by 2. (A different order can be used, but it is easy to divide by 10 first.)

$$\frac{\cancel{140}^{14}}{\cancel{560}_{56}} = \frac{\cancel{14}^{2}}{\cancel{56}_{8}} = \frac{\cancel{2}^{1}}{\cancel{8}_{4}} = \frac{1}{4}.$$

The fraction who have studied an OU course is $\frac{1}{4}$.

In the media, fractions of a group are sometimes expressed in a slightly different way. For example, in the last activity, $\frac{1}{4}$ of the people surveyed had studied an OU course. This is the same as saying: 'Out of the group surveyed, **one in four** people had studied an OU course.'

Activity 30: True or false?

Read the following statements and decide whether they are true or false.

(a) 'Only a fraction of the group were on time' means that less than half the people were on time. F

(b) You can write any fraction in a decimal form. T

(c) Fractions always have a value of less than one. F

(d) Brain stretcher – a number can always be written as a fraction. F

Comment

(a) False. For example, the fraction could be $\frac{3}{4}$ or $\frac{5}{8}$ which are both bigger than a half. However, in everyday language the statement does suggest that it is a small fraction, perhaps less than a half. You will find that several everyday words are also used in mathematics where they have a different or more precise meaning, for example the word 'product'. It is important that you use the correct interpretation, so adding these words to your dictionary might help.

(b) True. A fraction can be thought of as a division. For example $\frac{7}{8}$ is the same as $7 \div 8 = 0.875$. However some decimal fractions do not stop – they have a repeating set of digits such as: $\frac{2}{7} = 0.285714285714....$ These are known as recurring decimals and are written by placing a dot over the first and last digit of the repeating set:

$\frac{2}{7} = 0.\dot{2}8571\dot{4}$.

In practical situations, these decimal numbers can be rounded.

(c) False. Proper fractions in which the numerator is less than the denominator, do have a value less than one. However, improper fractions such as $\frac{7}{3}$ (seven-thirds) are greater than 1.

(d) False. Some numbers can be written as fractions. For example, whole numbers such as 8 can be written with a denominator of 1, so $8 = \frac{8}{1}$. If you have a finite decimal number such as 0.375 which stops after a certain number of decimal places, then it can also be written as a sum of fractions. For example: $0.375 = \frac{3}{10} + \frac{7}{100} + \frac{5}{1000}$.

These fractions can then be added together and cancelled (as you will see in the next section): $\frac{375}{1000} = \frac{3}{8}$.

Decimals which have a repeating sequence can also be written as fractions. For example: $0.\dot{3} = \frac{1}{3}$ and $0.\dot{3}\dot{6} = \frac{4}{11}$. However, it can be proved that there are some numbers such as $\sqrt{2}$, which cannot be written as a fraction. These are known as **irrational** numbers.

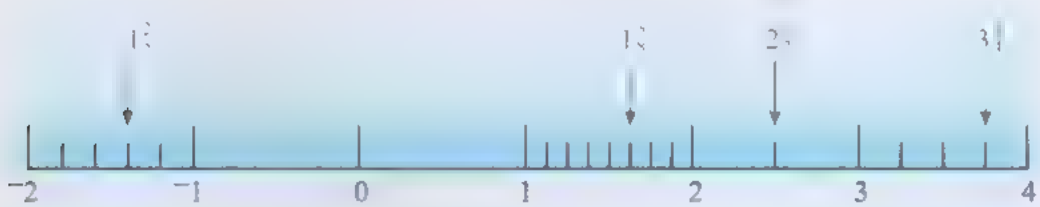
In most of the examples considered so far, you have been using fractions to describe part of a whole. However, they can be thought of as numbers in their own right as well. Mixed numbers have a whole number part and a proper fraction part, for example $2\frac{1}{3}$.

Activity 40 Fractions on the line

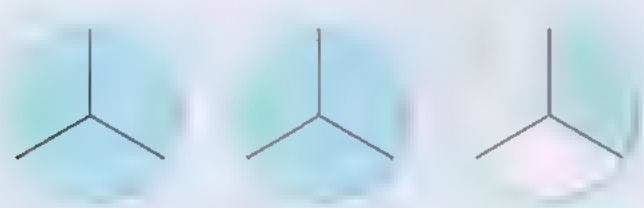
Draw a number line from $^{-}2$ to 5. Mark the following fractions on the number line: $2\frac{1}{2}, 3\frac{3}{4}, 1\frac{5}{8}$ and $^{-}1\frac{2}{5}$.

Comment

You can determine the position by dividing the length between the two appropriate whole numbers on the scale into 2, 4, 8 or 5 equal pieces as appropriate, and then marking the fractions on it as shown below.



Mixed numbers such as $2\frac{1}{3}$ can also be written as **improper** fractions in which the whole number part is turned into the same type of fraction as the fractional part, in this case thirds. You can imagine this in terms of pizzas – if you have two and a third pizzas and cut the two whole pizzas into thirds, how many thirds will you have?



In this case the total number of thirds will be $(2 \times 3) + 1 = 7$ and so

$$2\frac{1}{3} = \frac{(2\times3)+1}{3} = \frac{6+1}{3} = \frac{7}{3}$$

You can change improper fractions back into mixed numbers as well. For example, for $\frac{17}{8}$, imagine you have some pizzas cut into eighths and that you have 17 eighths or pieces. You know that eight eighths make one whole and that two lots of eight is 16. This would leave one-eighth over. So $\frac{17}{8} = 2\frac{1}{8}$.

Activity 41 Mixed numbers and improper fractions

(a) Change the following mixed numbers into improper (or top heavy) fractions.

(i) $3\frac{1}{4}$ (ii) $2\frac{4}{5}$ (iii) $7\frac{3}{8}$

(b) Change the following improper fractions into mixed numbers.

(i) $\frac{23}{5}$ (ii) $\frac{15}{7}$ (iii) $\frac{17}{4}$

Comment

(a) (i) There are 4 quarters in each whole, so 3 wholes will give 12 quarters. The extra quarter makes 13 overall so the fraction is $\frac{13}{4}$.

(ii) There are 5 fifths in each whole, so 2 wholes will give 10 fifths. The extra 4 fifths make 14 overall so the fraction is $\frac{14}{5}$.

(iii) There are 8 eighths in each whole, so 7 wholes will give 56 eighths. The extra 3 eighths makes 59 overall so the fraction is $\frac{59}{8}$.

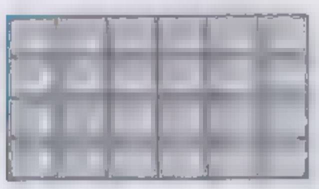
(b) (i) Since $4 \times 5 = 20$, 20 fifths will make up 4 wholes. This leaves 3 fifths over so the fraction is $4\frac{3}{5}$.

(ii) Since $2 \times 7 = 14$, 14 sevenths will make up 2 wholes. This leaves 1 seventh over so the fraction is $2\frac{1}{7}$.

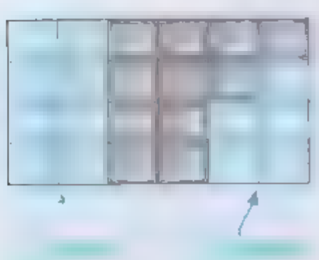
(iii) Since $4 \times 4 = 16$, 16 quarters will make up 4 wholes. This leaves 1 quarter over so the fraction is $4\frac{1}{4}$.

Now work through Section 3.1 of Chapter 3 of the Calculator booklet.

For the next example, it helps to have a large bar of chocolate with 24 squares. If your bar has 32 squares, break some off and eat them if you like!



This bar has 24 chocolate pieces. So each piece is $\frac{1}{24}$ of the bar and the whole bar is $\frac{24}{24}$ or 1. There are four identical rows, so each row is $\frac{1}{4}$ of the bar. There are also six identical columns, so each column is $\frac{1}{6}$ of the bar and two columns are $\frac{2}{6}$ or $\frac{1}{3}$ of the bar.



Now imagine that you eat a third of the bar and, still feeling peckish five minutes later, decide to eat a further five pieces. What fraction of the bar will you have eaten and, more importantly, what fraction is left?

The fraction eaten will be $\frac{1}{3} + \frac{5}{24}$, but it is quite difficult to imagine what fraction this is as two different kinds of fractions (thirds and twenty-fourths) are involved. However, if you change the fractions into the same type by thinking of $\frac{1}{3}$ as $\frac{8}{24}$, the fraction eaten is $\frac{8}{24} + \frac{5}{24} = \frac{13}{24}$. You can check this on the bar of chocolate!

The fraction that is left will be $1-\frac{13}{24}$. So, thinking of the whole bar as $\frac{24}{24}$, the calculation becomes $\frac{24}{24}-\frac{13}{24}=\frac{11}{24}$. This method of changing the fractions into the same kind using equivalent fractions can be used for adding and subtracting fractions in general. If you need to add or subtract fractions, the first question you need to ask yourself is 'Are these the same kind of fractions?'. If the answer is no, then you need to convert the fractions into equivalent fractions which are of the same type.

Activity 42 Adding and subtracting fractions

Work out the following.

(a) $\frac{5}{18}+\frac{7}{18}$ (b) $\frac{5}{6}+\frac{3}{4}$ (c) $2\frac{2}{3}+3\frac{1}{8}$

(d) $\frac{15}{16}-\frac{5}{16}$ (e) $\frac{19}{24}-\frac{3}{8}$ (f) $4\frac{1}{4}-2\frac{5}{8}$

Comment

(a) Both the fractions are eighteenths, so they can be added together directly. Adding 5 eighteenths and 7 eighteenths gives 12 eighteenths. This can be written as $\frac{5}{18}+\frac{7}{18}=\frac{5+7}{18}=\frac{12}{18}=\frac{2}{3}$. Notice how $\frac{12}{18}$ has been cancelled to $\frac{2}{3}$, by dividing the top and the bottom by 6.

(b) This sum involves sixths and quarters which are different types of fraction. However, you can change both into twelfths since $\frac{5}{6}=\frac{10}{12}$ and $\frac{3}{4}=\frac{9}{12}$. So the sum is $\frac{5}{6}+\frac{3}{4}=\frac{10}{12}+\frac{9}{12}=\frac{19}{12}=1\frac{7}{12}$.

(c) Here you can add the whole numbers together first, $2+3=5$, and then add the fractions together by changing them into twenty-fourths. So the sum is $2\frac{2}{3}+3\frac{1}{8}=5+\frac{2}{3}+\frac{1}{8}=5+\frac{16}{24}+\frac{3}{24}=5\frac{19}{24}$.

(d) Both the fractions are sixteenths. Subtracting 5 sixteenths from 15 sixteenths will leave 10 sixteenths which can be cancelled to five-eighths by dividing top and bottom by 2. This can be written as $\frac{15}{16}-\frac{5}{16}=\frac{10}{16}=\frac{5}{8}$.

(e) This calculation involves twenty-fourths and eighths which are different types of fraction. You can convert 3 eighths into 9 twenty-fourths by multiplying top and bottom of the fraction by 3. So the calculation becomes $\frac{19}{24}-\frac{3}{8}=\frac{19}{24}-\frac{9}{24}=\frac{10}{24}=\frac{5}{12}$. Note the answer is reduced to its lowest form by dividing the top and the bottom by 2.

(f) This one is a bit trickier, but there are several ways to tackle it. If it helps, think in terms of pizzas – you have $4\frac{1}{4}$ pizzas and need to take $2\frac{5}{8}$ away for your friend. What would you do? Well you could take the 2 pizzas away first, then take $\frac{5}{8}$ of one of the whole pizzas (as $\frac{5}{8}$ is a bigger portion than $\frac{1}{4}$). This would leave one whole pizza, $\frac{3}{8}$ of one pizza and the $\frac{1}{4}$ (or $\frac{2}{8}$) pizza you had not touched yet, which is $1\frac{5}{8}$ pizzas overall. We can write this mathematically as

$$4\frac{1}{4}-2\frac{5}{8}=4\frac{1}{4}-2-\frac{5}{8}=2\frac{1}{4}-\frac{5}{8}=1\frac{1}{4}+(1-\frac{5}{8})=1\frac{2}{8}+\frac{3}{8}=1\frac{5}{8}.$$

Alternatively, separate the $4\frac{1}{4}$ pizzas into $1\frac{1}{4}$ and 3 pizzas. Take away $2\frac{5}{8}$ from the 3 whole ones leaving $\frac{3}{8}$ of one pizza. Then add this to the $1\frac{1}{4}$ pizzas to give $1\frac{5}{8}$. We can write this mathematically as

$4\frac{1}{4} - 2\frac{5}{8} = 1\frac{1}{4} + 3 - 2\frac{5}{8} = 1\frac{1}{4} + \frac{3}{8} = 1\frac{5}{8}$.

Using fractions in real life

Why do you think the authors used fractions of the group (rather than the numbers of people) to report their results?

In 2001, The New Policy Institute (whose mission is to advance social justice in a market economy) published a report called 'Young people, financial responsibilities and social exclusion'. The report summarised the results of a survey on 210 young people, two thirds of whom lived in YMCA hostels. Of the young people who were surveyed, half were in debt and, of these, a third owed more than £1000.

Activity 4[illegible] Young people and debt

(a) Of the young people who participated in the survey, how many lived in the YMCA hostels?

(b) How many of the young people were in debt?

(c) Use your answer to work out how many people owed more than £1000.

(d) What fraction of the total number of people surveyed owed more than £1000?

Comment

(a) One-third of the group is $210 \div 3 = 70$ people. So $\frac{2}{3}$ of the group will be 2 lots of 70 people or 140 people.

(b) Dividing 210 by 2 to find half the group, shows that 105 young people were in debt.

(c) One-third of the group of 105 people owed more than £1000. As $105 \div 3 = 35$, 35 people owed more than £1000.

(d) Since 210 people were surveyed, the fraction who owed more than £1000 is $\frac{35}{210}$, which cancels down to $\frac{1}{6}$.

Part (d) is not a surprising result. One-third of a half is indeed a sixth. To convince yourself, take a piece of A4 paper. Fold it in half along the long side and shade one half of the paper. Then fold it into thirds along the short side. The paper is now split into six equal pieces – sixths. If you look at the shaded half, you can see that one third of this portion is a sixth as shown below.

Similarly if we wanted to find the fraction of young people who were in debt but owed £1000 or less, the fraction would be two-thirds of one-half. From the diagram, this is two-sixths, which cancels to one third. Mathematically, this is written as $\frac{2}{3} \times \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$.

To multiply two fractions together, it looks as if you multiply the numerators together and then the denominators together and finally cancel the answer. In fact this is true generally, although it is usually easier to cancel before multiplying out. For example, multiplying the numerators together and then multiplying the denominators together gives

$$\frac{2}{3} \times \frac{1}{2} = \frac{2 \times 1}{3 \times 2}.$$

Now dividing both the numerator and the denominator by 2 gives

$$\frac{\cancel{2}^{1} \times 1}{3 \times \cancel{2}_{1}} = \frac{1 \times 1}{3 \times 1} = \frac{1}{3}.$$

This calculation is equivalent to cancelling at the start of the calculation:

$$\frac{\cancel{2}^{1}}{3} \times \frac{1}{\cancel{2}_{1}} = \frac{1 \times 1}{3 \times 1} = \frac{1}{3}.$$

The box below summarises the procedure.

Multiplying fractions

For example, $2\frac{3}{4} \times 1\frac{1}{3}$

Write mixed numbers as improper fractions first. Cancel as far as possible. $2\frac{3}{4} \times 1\frac{1}{3} = \frac{11}{\cancel{4}_{1}} \times \frac{\cancel{4}^{1}}{3}$

Multiply the numerators together and then multiply the denominators together. $= \frac{11}{1} \times \frac{1}{3}$

$= \frac{11 \times 1}{1 \times 3} = \frac{11}{3}$

Write as a mixed number, if appropriate. $= 3\frac{2}{3}$

Activity 44 Multiplying fractions

What number gives the same answer whether you add $\frac{1}{2}$ to it or multiply it by $\frac{1}{2}$?

In the New Policy Institute survey, $\frac{3}{5}$ of the 210 young people had a bank account. Of these, $\frac{1}{3}$ had an overdraft facility and approximately $\frac{2}{5}$ had a cheque book. How many people had an overdraft? How many had a cheque book?

Comment

One-fifth of the group is $210 \div 5 = 42$ people.

So $\frac{3}{5}$ of the group will be $3 \times 42 = 126$ people. So 126 people have a bank account. A third of this group is $126 \div 3 = 42$ people, so 42 people have an overdraft facility. One-fifth of the group is $126 \div 5 = 25.2$ people. So two-fifths will be $2 \times 25.2 = 50.4$ people. This rounds to 50, so about 50 people have a cheque book. Alternatively the number of people with a cheque book is $\frac{3}{5} \times \frac{2}{5} \times 210$.

210 can be written as $\frac{210}{1}$ since $210 \div 1 = 210$.

Since 210 and 5 can both be divided by 5, the calculation can then be cancelled down as follows:

$$\frac{3}{5} \times \frac{2}{\cancel{5}_1} \times \frac{\cancel{210}^{42}}{1} = \frac{3 \times 2 \times 42}{5 \times 1} = \frac{252}{5} = 50.4.$$

This is approximately 50 people. Since the fractions are only approximate, you need to interpret the results of your calculations carefully, particularly if they involve fractions of a person!

In Activity 44, you needed to multiply fractions together and you saw two different approaches. In the first you calculated the actual number of people at each stage, while in the second you multiplied the fractions together first and then calculated the number of people the fraction represented. You can use whichever method you feel is easier.

Dividing fractions is a little more difficult, so we'll try out two different approaches: one using your calculator so that you can get a feel for what's happening, and another by considering a simple example first and seeing if we can use a similar strategy.

Now work through Section 3.2 of Chapter 3 of the Calculator Booklet.

From the calculator exercises, you can see that to divide by a fraction, you turn the fraction upside down and multiply by it instead. But why does that work?

Let's start by considering something easier first, say $6 \div 2$.

This can be written as the fraction $\frac{6}{2}$. If we cancel it by dividing the top and the bottom by 2, we get $\frac{3}{1}$ which is 3. That might seem a complicated way of dealing with a sum like $6 \div 2$, but it does give us a clue on how to tackle a problem that involves dividing by a fraction. What we have done here is to rewrite the fraction $\frac{6}{2}$ in a form that has a denominator of 1, because we know we can work out the value of such a fraction easily. So let's try the same approach with a more difficult problem: $6 \div \frac{2}{5}$.

This 'triple' fraction looks very complicated, but just carry on.

Following the same approach as before, we first write this as a fraction: $\frac{6}{\frac{2}{5}}$.

We know that we want to try to rewrite this as a fraction with a denominator of 1 and we also know that we can multiply the top and the bottom of a fraction by the same number without changing its value. So we need to choose a number, so that when we multiply $\frac{2}{5}$ by it, we get 1. Whatever number we pick will have to cancel with the 2 on the top of the fraction and with the 5 on the bottom, so try $\frac{5}{2}$. Now see what happens if we multiply top and bottom of the fraction by $\frac{5}{2}$. The denominator then works out to be 1:

$$\frac{6}{\frac{2}{5}} = \frac{6\times\frac{5}{2}}{\frac{\cancel{2}^1}{\cancel{5}_1}\times\frac{\cancel{5}^1}{\cancel{2}_1}} = \frac{6\times\frac{5}{2}}{1} = 6\times\frac{5}{2} = \frac{\cancel{6}^3}{1}\times\frac{5}{\cancel{2}_1} = \frac{3\times5}{1\times1} = 15.$$

So dividing a number by $\frac{2}{5}$ is the same as multiplying the number by $\frac{5}{2}$.

$$6 \div \frac{2}{5} = 6\times\frac{5}{2} = \frac{\cancel{6}^3}{1}\times\frac{5}{\cancel{2}_1} = \frac{3}{1}\times\frac{5}{1} = \frac{15}{1} = 15.$$

This process of turning the fraction **after** the division sign upside down and then multiplying by it, can be used in any division problem.

Dividing fractions

For example, $2\frac{4}{9} \div 1\frac{1}{3}$

Write as improper fractions first.	$2\frac{4}{9} \div 1\frac{1}{3} = \frac{22}{9} \div \frac{4}{3}$
Turn the fraction after the ÷ sign upside down and change the ÷ to ×.	$= \frac{22}{9}\times\frac{3}{4}$
Cancel if possible and multiply out.	$\frac{\cancel{22}^{11}}{\cancel{9}_3}\times\frac{\cancel{3}^1}{\cancel{4}_2} = \frac{11}{6}$
Write as a mixed number, if appropriate.	$= 1\frac{5}{6}$.

Activity 4[illegible]: Dividing fractions

(a) Work out the following.

(i) $4 \div \frac{2}{3}$

(ii) $2\frac{2}{3} \div \frac{7}{15}$

(b) If it takes $\frac{3}{4}$ of an hour to clean a car, how many cars can be cleaned in a $7\frac{1}{2}$ hour day?

(c) One side of a garden measures $12\frac{1}{2}$ m. A fence made of $1\frac{1}{2}$ m wide panels is to be put up along this side. How many panels will be needed?

Comment

(a) (i) $4 \div \frac{2}{3} = 4 \times \frac{3}{2} = \frac{\cancel{4}^{2}}{1} \times \frac{3}{\cancel{2}_{1}} = \frac{2 \times 3}{1} = 6.$

(ii) Change the first fraction into an improper fraction before dividing:

$$2\tfrac{2}{3} \div \tfrac{7}{15} = \frac{8}{\cancel{3}_1} \times \frac{\cancel{15}^{5}}{7} = \frac{8 \times 5}{1 \times 7} = \frac{40}{7} = 5\tfrac{5}{7}.$$

For practical problems, you can change the fractions into decimals first, if you prefer.

(b) Here we need to find how many lots of 'three quarters' there are in 'seven and a half', so the calculation is $7\frac{1}{2} \div \frac{3}{4} = \frac{15}{2} \div \frac{3}{4} = \frac{\cancel{15}^{5}}{\cancel{2}_1} \times \frac{\cancel{4}^{2}}{\cancel{3}_1} = 10.$
Hence 10 cars can be cleaned in a day.

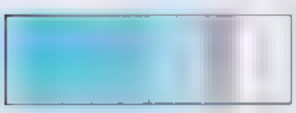
(c) We need to find how many lots of $1\frac{1}{2}$ m there are in $12\frac{1}{2}$ m.
So the calculation is $12\frac{1}{2} \div 1\frac{1}{2} = \frac{25}{2} \div \frac{3}{2} = \frac{25}{\cancel{2}_1} \times \frac{\cancel{2}^{1}}{3} = \frac{25}{3} = 8\frac{1}{3}.$
Hence, eight panels will be too short and nine panels are needed.

Now work through Section 3.3 of Chapter 3 of the Calculator booklet.

The next activity is a Brain stretcher. Have a go and if you get stuck, don't worry, just make a note of the things that you try to do to get unstuck – try to use some of the ideas on 'Reading mathematics' in Section 2.8. This will be good preparation for the next section which discusses how to get yourself unstuck in more detail!

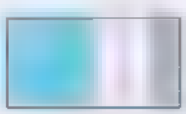
Activity 48 Brain stretcher: Egyptian fractions

The ancient Egyptians only used the fraction $\frac{2}{3}$ and unit fractions. Unit fractions are fractions whose numerator is 1, e.g. $\frac{1}{4}$ and [illegible]. They expressed all other fractions, by adding different unit fractions together.
For example, $\frac{4}{5}$ could be written as $\frac{1}{2} + \frac{1}{5} + \frac{1}{10}$.



Check on your calculator or on a diagram.

(a) Can you express $\frac{3}{4}$ and $\frac{7}{8}$ as the sum of unit fractions? How did you work these out?
Now, using your calculator, check that $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$.



If you divide each of $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ and 1 by 2, you should find that $\frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$. We know that $\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$. So, replacing the $\frac{1}{2}$ by $\frac{1}{4} + \frac{1}{6} + \frac{1}{12}$, an alternative expression for $\frac{4}{5}$ in terms of unit fractions is:
$\frac{4}{5} = (\frac{1}{4} + \frac{1}{6} + \frac{1}{12}) + \frac{1}{5} + \frac{1}{10}.$

Since the fractions can be added together in any order, this can be rearranged to $\frac{4}{5} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10} + \frac{1}{12}$. (You might like to check it on your calculator!)

(b) By writing $\frac{1}{10}$ as $\frac{1}{20} + \frac{1}{30} + \frac{1}{60}$, write down another set of unit fractions whose sum is $\frac{4}{5}$.

(c) How many ways do you think a proper fraction can be expressed as a sum of unit fractions?

Finding the **fewest** number of unit fractions that can be added together is a more challenging problem!

Comment

(a) $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$ and $\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$. One way to work these out is to use a diagram split into quarters or eighths, with the required fraction shaded. Alternatively you can think about the problem practically: $\frac{3}{4}$ means '3 divided by 4'. So imagine three bars of chocolate being shared equally among four people. If you broke all the bars in half and gave one half to each person, there would still be two half bars left. These could be broken in half again and a quarter then given to each person. So each person would get a half and a quarter.

(b) Replacing $\frac{1}{10}$ in the original expression gives: $\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{20} + \frac{1}{30} + \frac{1}{60}$.
Alternatively, you may have replaced the $\frac{1}{10}$ in the expression above to get:
$\frac{4}{5} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + (\frac{1}{20} + \frac{1}{30} + \frac{1}{60}) + \frac{1}{12} = \frac{4}{5} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{60}$.

(c) You can always replace the smallest unit fraction (i.e. the fraction with the largest denominator) in the sum by the sum of three smaller fractions and this process can be applied indefinitely so there are infinitely many different ways of splitting a fraction into a sum of unit fractions.

3.2 What to do when you get stuck!

The last activity and the margin note contained some difficult ideas and you may have found these quite challenging. As you discovered in Chapter 1, everyone gets stuck at some point when they are doing maths, so it is important to have some strategies for tackling problems and getting yourself going again, whether you are working on a real-life problem or tackling something more abstract. Even simple things like discussing the problem with someone else or having a break can help, if it enables you to come back to your problem with a fresh and open mind. In the example above, drawing a diagram or thinking about the problem practically might have helped you too. You have also met the idea of the mathematical modelling cycle as a way of approaching real-life problems. In this section, we are going to look at some further strategies by considering the following problem.

> My father has recently moved into an old cottage to be closer to us and it needs a lot of renovating, including putting some insulation in the loft. Although I think it's going to be cheaper to put the

insulation in myself, he can get a grant from the District Council if the insulation is installed by an approved contractor and that work would be guaranteed.

So which is the better option?

Step 1: Do you understand the problem?

The first step in any mathematics is to check that you understand the problem – can you explain it in your own words? Here the problem is clear, but what do you think is meant by the 'better' option? For the moment, I'll concentrate on costs and interpret this as the 'cheaper' option.

Then it is often helpful to write down **what you know** and **what you want** to find. This can include both information and mathematical techniques that you think might be relevant. After measuring the floor space, visiting a DIY store and contacting the District Council, I wrote down the following lists.

I know:	I want:
1. The loft is approximately 6.75 m by 7.5 m 2. A grant of £250 is available 3. Rolls of 170 mm thick insulation are 370 mm wide and 4.7 m long 4. The rolls cost £5.99 each but there is a '3 for 2' offer	To find the cheapest option

Activity 47 Anything else?

Do I need any further information? What is it?

Comment

Yes, I still need to get some quotes from approved contractors. The cheapest quote is £400. I also need to check how to install the insulation – will I need to buy any extra equipment? The instructions recommend wearing a face mask and gloves (an additional £15). The insulation has to be fitted between the joists, leaving an air gap of 50 mm at the edges of the roof and must not be put under the water tank. Fortunately, a quick check in the loft shows that the insulation fits perfectly between the joists, apart from under the eaves where it will need to be trimmed.

Step 2: Tackling the problem

This is quite a complicated and involved problem, so a useful strategy here would be to **break it down into more manageable chunks**, by concentrating on one bit at a time. There are two separate problems.

- How much would my father have to pay if he got the £250 grant and used the contractor?
- How much would the DIY option cost?

The second problem can be broken down further as well – how much insulation will I need and how much will it cost?

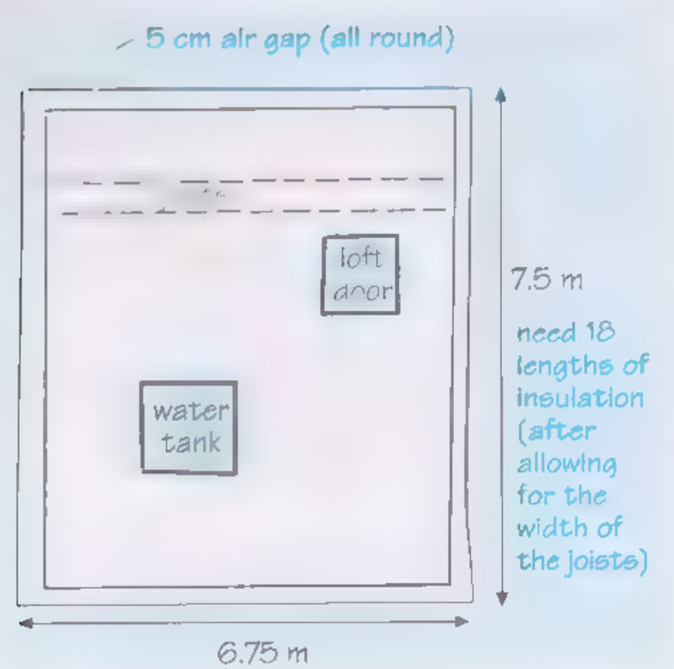
Activity 40 The contractor's costs

How much will it cost if the contractor installs the insulation and the grant is used?

Comment

The contractor charges £400 and the grant is £250. So the overall cost will be £400 – £250 = £150.

So returning to the DIY option, I now need to work out how many rolls of insulation are needed in the loft. It is a good idea at this stage to think back and see if you have ever seen a problem anything like this before and, if so, whether the approach you used there could also be used successfully here. In this case, the wallpaper problem in Chapter 2 is similar and in that case **drawing a diagram** did help. In fact, drawing a diagram or visualising a problem in some other way is a useful strategy to use in many problems. Here, after counting how many joists there were in the loft, I sketched out a rough plan.



The loft door and the top and sides of the tank will be covered with insulation.

From the diagram, to find the length of insulation needed between each pair of joists, subtract the length allowed for the air gaps from the length of the loft. The air gap at each end is 50 mm.

$50 \text{ mm} = 50 \div 1000 \text{ m} = 0.05 \text{ m}$.

So the length of insulation needed to put between two joists is $6.75 - 2 \times 0.05 = 6.65$ m.

An extra 3.7 m of insulation is needed to go around the water tank.

Activity 4 How much insulation?

Work out the length of insulation needed and how many rolls to buy.

Comment

I need 18 lengths of 6.65 m insulation plus an extra 3.7 m to go around the water tank.

So the total length of insulation required is $18 \times 6.65 \text{ m} + 3.7 \text{ m} = 123.4$ m.

Each roll contains a length of 4.7 m. So the total number of rolls is $123.4 \div 4.7 = 26.26$ (2 d.p.). So rounding this number up, I will need 27 rolls. Notice that it does not matter if I cannot get a whole length from a roll as I can cut the extra bit from a new roll.

The final step is to work out the cost with the '3 rolls for the price of 2' offer. So if we put the rolls into groups of three, there will be nine groups (since $9 \times 3 = 27$). Each roll costs approximately £6, so each group of three will cost about £12 and nine groups will cost about £108. The mask and gloves will cost an extra £15.

Activity 5 The costs of DIY

Make a quick estimate of the total cost of the DIY option using some of the mental strategies from Chapter 2. Then calculate the amount accurately on your calculator. Would you pay the contractor or do the job yourself?

Comment

The total cost is roughly £108 + £15. Splitting the £15 into £10 + £5, the calculation becomes £108 + £10 = £118 and £118 + £5 = £123. On the calculator, the total cost is 9 × 2 × £5.99 + £15 = £122.82 which agrees with the estimate.

This is £27.18 less than the cost of using the contractor, so the DIY option is cheaper. However is this the best option for me overall? Strictly I would need to allow some extra money for petrol to pick up the rolls from the DIY store

– that could be several journeys as the rolls are bulky and my car is small. Also it would take me quite a lot of time to buy and put the insulation in by myself and my work is not guaranteed. So, all things considered, I decided to opt to pay for the contractor, and use the time I would save to start decorating the bedroom! Although the mathematical solution suggests the DIY option, within the context of the problem as a whole, using the contractor is the better option for me. Using maths has helped me to make the decision, but other considerations have also played a part.

Step 3: Looking back

With any problem, it is always worth thinking back over how you tackled the problem and whether you could use any of the ideas for problems in the future. Here breaking the problem down into small steps, using a diagram and thinking back over similar problems helped. These strategies could be used again, particularly if the area to be covered was more complicated. It is also worth thinking how the problem ties in with what you already know. For example, this is a real-life problem, so it should be possible to use the modelling cycle to tackle it – can you identify those different stages here? (Compare it with the wallpaper problem if you get stuck.) Are there any other ways you could have tackled the problem? Can you check your result and does it seem reasonable? For example, if you knew how to tackle the area of the loft, you may have preferred that as an approach – do your two answers agree and if not, why not?

This problem involved quite a lot of different mathematical techniques – rounding, estimating, and using your calculator. Would further practice on these techniques help you to tackle problems more confidently in future?

Although we have used a real problem to illustrate these strategies, they work just as well on more abstract problems. Another important strategy is discussing the problem with someone else. By explaining how you have tried to tackle the problem in your own words, you might clarify your thinking sufficiently to see a way forward. Alternatively, the person may be able to suggest some of their own ideas which might help. However, even if the other person just asks questions like: ‘But why have you done that calculation there?’ or ‘I don’t see how you’ve got that …’, these questions might challenge you to think of another approach or to justify and check your results so far. Trying to teach someone else is a very good way of learning and sorting out your own ideas thoroughly!

Activity 7 Stuck – what do you do?

Think back over the course so far and also the occasions when you have used maths in real life. Try to remember a situation where you got stuck with some mathematics. What did you try to do to resolve the situation? What strategies have you already met in the course? Make a list (preferably on a piece of card so that you can refer to it easily) of the strategies you can use when tackling a problem.

Comment

You probably have several strategies from your own mathematical experiences already.

For example, a student nurse made the following comment.

> I'm still not sure about working out the more complicated drug doses, so I check the answer is similar to previous doses and then I discuss the calculation and result with the pharmacist.

Some of the other strategies are summarised in the box below – feel free to add your own!

Stuck?

- Check you understand the problem.
- Write down what you know – this includes all the information you have, any techniques that might be helpful and any of your ideas for tackling the problem.
- Write down what you want. Can you put the question in your own words? Write down other questions as well; for example: 'What does ... mean?', 'How are these ideas linked?'.
- Make a note of any ideas that occur to you; such thoughts might start 'I wonder if...' or 'Maybe ... might work...'.
- Break the problem down into smaller steps which you can tackle one at a time.
- Have you seen something like this before? Could you use a similar approach?
- Draw a diagram or use a physical model.
- Can you solve a simpler problem first? For example, by temporarily replacing awkward numbers by smaller or simpler ones?
- If it is an abstract problem, would looking at an example with numbers or a real context help?
- Check any ideas you've had and go through your working carefully – an arithmetical slip or other mistake can stop you from moving ahead! Is your answer reasonable? Can you check it some other way?
- Have a break – a walk, a cup of coffee or some household chores!
- Discuss the problem with someone else – another student, your tutor, family or friends.

3.3 Percentages

One type of fraction that is used a lot in everyday life including in the media, are those fractions with a denominator of 100 such as $\frac{25}{100}$ or $\frac{50}{100}$. These fractions are so useful that they use a simpler notation and are usually referred to as percentages. So $\frac{25}{100}$ is written as 25% which is read as 'twenty-five per cent' and means '25 out of a hundred'. One of the reasons why percentages are useful is that you can compare proportions easily. For example if a restaurant noted that 42% of customers opted for a meat dish, 35% opted for a vegetarian dish and 23% opted for fish, you can see straightaway that the meat dish was most popular. This is much easier than comparing the corresponding fractions of $\frac{21}{50}$, $\frac{7}{20}$ and $\frac{23}{100}$.

In fact, fractions, decimals and percentages are all interchangeable, so you can choose to use whichever is most appropriate for your situation. This also means that if you have already got a good understanding of fractions and decimals, you should be able to cope with percentages fairly easily too. From the definition of a percentage, 75% means $\frac{75}{100}$; this cancels to $\frac{3}{4}$. Alternatively, $\frac{75}{100}$ means $75 \div 100$ or 0.75. This technique can be summarised as follows.

> To change a percentage to a fraction or decimal, divide by 100.

Now how can you write a fraction or a decimal as a percentage? If you are not sure, try using the I know I want strategy – here you know the fraction and you want the percentage. What does percentage mean? It means writing the fraction with a denominator of 100. How can you do that? How do you change one type or fraction into another? Where have you seen this before? Hopefully you'll recognise this as an equivalent fraction problem and multiplying top and bottom by 100, should work. Try using a particular numerical example to check:

$$\frac{2}{5} = \frac{2 \times \cancel{100}^{20}}{\cancel{5}_1 \times 100} = \frac{40}{100} = 40\%$$

Multiplying the top and the bottom by 100 is the same as multiplying the fraction by 100%. This method can be summarised as follows.

> To change a fraction or decimal to a percentage, multiply by 100%.

For example: $0.725 = 0.725 \times 100\% = 72.5\%$ and

$\frac{5}{8} = \frac{5}{8} \times 100\% = \frac{500}{8}\% = 62.5\%$.

Activity [illegible] Fractions, decimals and percentages

(a) Write the following percentages as fractions and decimals.

(i) 10% (ii) 25% (iii) 50% (iv) 125% (v) 0.5%

(b) Write the following as percentages.

(i) $\frac{3}{4}$ (ii) $\frac{5}{8}$ (iii) $\frac{1}{3}$ (iv) 0.4 (v) 0.0075 (vi) 1.2

(c) A radio programme reported: 'Our "Going Solo" survey of 4000 single people found that only 1 in 5 is happy on their own.' Write '1 in 5' as a fraction, a decimal and a percentage. Which form do you think is easiest to understand?

Comment

(a) To change a percentage to a fraction or decimal, divide by 100.

(i) $10\% = \frac{10}{100} = \frac{1}{10}$ or $10\% = 10 \div 100 = 0.1$.

(ii) $25\% = \frac{25}{100} = \frac{1}{4}$ or $25\% = 25 \div 100 = 0.25$.

(iii) $50\% = \frac{50}{100} = \frac{1}{2}$ or $50\% = 50 \div 100 = 0.5$.

(iv) $125\% = \frac{125}{100} = \frac{5}{4}$ or $125\% = 125 \div 100 = 1.25$.

(v) $0.5\% = \frac{0.5}{100} = \frac{1}{200}$ or $0.5\% = 0.5 \div 100 = 0.005$.

(b) Multiplying each fraction or decimal by 100% gives the following answers.

(i) $\frac{3}{4} = \frac{3}{4} \times 100\% = 75\%$.

(ii) $\frac{5}{8} = \frac{5}{8} \times 100\% = 62.5\%$.

(iii) $\frac{1}{3} = \frac{1}{3} \times 100\% = 33\frac{1}{3}\%$.

(iv) $0.4 = 0.4 \times 100\% = 40\%$.

(v) $0.0075 = 0.0075 \times 100\% = 0.75\%$.

(vi) $1.2 = 1.2 \times 100\% = 120\%$.

(c) '1 in 5' is the same as $\frac{1}{5}$ which is the same as $1 \div 5 = 0.2$ or $\frac{1}{5} \times 100\% = 20\%$.

The report could have said 'A fifth' or '20%' of people in the survey, but '1 in 5' makes the proportion easy to visualise for people not comfortable with fractions and percentages.

You have already seen how using percentages can make comparisons easier than using fractions. So that leaves one question: how do you express one quantity as a percentage of another? For example, if 42 people out of a group of 70 people agree to participate in a new community project, what percentage of the group is this?

You can use the I know / I want strategy again here. You know how to express one number as a fraction of another and you also know how to convert a fraction to a percentage, so putting these two processes together gives the following approach.

> To express one number as a percentage of another, divide the first number by the second and multiply by 100%

So in this case, cancelling by 10 and then by 7 gives

$$\frac{42}{70}\times 100\% = \frac{42}{\cancel{70}_7}\times\frac{\cancel{100}^{10}}{1}\% = \frac{\cancel{42}^6}{\cancel{7}_1}\times\frac{10}{1}\% = \frac{6\times 10}{1\times 1} = 60\%.$$

Hence, 60% of the group agree to participate in the community project. If you prefer, you can use your calculator to work out $42 \div 70 \times 100$.

[illegible]

An average Briton will spend £1 537 380 during his or her lifetime, a survey from insurer Prudential suggests. Housing, food and clothes are the biggest expenses, costing £552 772 on average during a lifetime. The next biggest outgoing is tax, with the average person paying £286 311 in income and council taxes. Leisure and going out cost £236 312 on average.

(BBC News, 2005)

What percentage of the 'average' Briton's lifetime expenditure will be spent on the following?

(a) Housing, food and clothes

(b) Tax

(c) How would you estimate how much you might spend over your lifetime? What assumptions would you make?

(d) The journalist decided to report the actual amounts spent on the different categories, rather than the percentages. Why do you think this was done?

Comment

(a) The percentage spent on housing, food and clothes is

$$\frac{552\,772}{1\,537\,380}\times 100\% \approx 36\% \text{ (to 2 s.f.)}.$$

(b) The percentage spent on taxes is $\frac{286311}{1537380} \times 100\% \approx 19\%$ (to 2 s.f.).

(c) At first sight it is difficult to believe that the average Briton will spend over £1.5 million and estimating how much you will spend over your lifetime is very difficult too, as there are so many things which can change. Assumptions that you might make are how long you will live, what income you might have over these years, how that income might change, and so on. For example if you assume that an average Briton is currently earning £20000 a year then in 45 years of working they would earn 45 × £20000 = £900000. But this does not allow for the rises in income over the years. For example, a new teacher in 1978 was earning around £3000 a year, whereas in 2005 they could expect to earn £20000.

(d) By reporting the numbers rather than the percentages, your attention is drawn to the very large sums involved straightaway and this has more impact than saying 'The average Briton will spend 36% of his money on food, housing and clothes'. However the amounts given for the different categories have been quoted to the nearest pound, which seems inappropriate, given the uncertainties in the future. It is also not clear what is meant by the 'average' Briton. Key questions here are how has the data been collected and what assumptions have been made in deriving these results?

Notice that, in the last two activities, to express one number as a percentage of another, you used two techniques that you were already familiar with: how to express one quantity as a fraction of another and then how to express a fraction as a percentage. This is a useful strategy in mathematics – if you are not sure how to work out a problem, try to change it into one that you do!

For example, suppose a mathematics exam is marked out of 75 and students need at least 40% to pass. Can you work out the pass mark?

The pass mark is 40% of 75. Here we can change the percentage into a fraction and then calculate $\frac{40}{100} \times 75$ to find the pass mark. Either cancel the fraction or multiply 40 by 75 and divide by 100 to show that the pass mark is 30.

This example illustrates how to find a given percentage of any number.

> To find a given percentage of a number, change the percentage to a fraction (or decimal) and multiply by the number.

Activity 54 Your pension fund

Read the extract below (from a television programme) and highlight what you think is the key information. How much should a 30-year-old earning £16000 a year and a 40-year-old earning £50000 be putting into their pension funds, according to this advice?

If you are working, do you know how much money you should be putting into your pension fund? Is it enough?

> Have you got enough money in your pension fund? If you start your pension at age 30 years, you need to put 15% of your income into your pension fund. If you start the fund at age 40 years, you need to put in 20% of your income. No one seems to be putting enough money into his or her pension fund.

(GMTV, 2005)

Comment

A 30-year-old starting a pension fund and earning £16000 a year would need to be putting in:

15% of £16000 $= \frac{15}{100} \times$ £16000 = £2400 a year. This would be £200 a month.

A 40-year-old starting a pension fund and earning £50000 a year would need to be paying in 20% of £50000 $= \frac{20}{100} \times$ £50000 = £10000 a year. This is approximately £833 a month.

It is a good idea to start paying into a pension fund as soon as you can.

Percentages are used a lot in business transactions, e.g. for calculating extra charges such as taxes or service charges, working out discounts on shopping, or specifying interest on a loan or on savings. For example, value added tax (VAT) is currently 17.5% for most goods (some have a reduced or zero rate), and some restaurants have a service charge of 15%. To calculate the total price, you can work out the extra charge (or discount) and then add (or subtract) it from the original price.

For example, to work out the VAT on a £48 item, you will need to calculate 17.5% of £48. Making a rough estimate first, 17.5% is approximately 20% and £48 is approximately £50.

Now 10% of £50 is £5, so 20% will be two lots of £5 or approximately £10. This makes the total price approximately £60.

Using the calculator, 17.5% of £48 = 17.5 ÷ 100 × £48 = £8.40.

This agrees fairly well with the estimate.

So the total price of the item including VAT is £48 + £8.40 = £56.40.

[illegible]

How would you calculate the final bill if you were charged VAT at 17.5% on an item costing £26?

Comment

Making an estimate first, 20% of £30 is approximately 2 × £3 or £6. So we expect the VAT to be less than £6. Using a calculator, 17.5% of £26 = $\frac{17.5}{100}$ × 26 = £4.55, which is less than £6, so the answer for the VAT looks reasonable.

Hence the total price is £26 + £4.55 = £30.55.

Alternatively you can tackle this using the mental strategies in Chapter 2.

17.5% = 10% + 5% + 2.5%.

10% of £26 = £2.60.

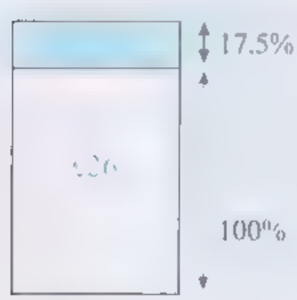
5% of £26 = £1.30 (half of £2.60).

2.5% of £26 = £0.65 (half of £1.30).

So 17.5% of £26 = £2.60 + £1.30 + £0.65 = £4.55.
The final bill would be the original cost plus the VAT = £26 + £4.55 = £30.55.

Another way of looking at this problem is to note that the total cost is the sum of the original cost and the VAT. The original cost can be thought of as 100% of the original cost since 100% = 1. Adding on the VAT, the total cost will be (100 + 17.5)% or 117.5% of the original cost.

So the total cost is 117.5% of £26 = $\frac{117.5}{100}$ × £26 = £30.55, as before.



Alternatively from the diagram, 1% is one-hundredth of £26 or £0.26. So 117.5% will be 117.5 lots of £0.26 which is 117.5 × £0.26 = £30.55.

Activity [illegible] Mail-order shopping

A mail-order company sends you some promotional material which says that as a valued customer, you are entitled to 15% off your next order. If you decide to buy some items costing £45, how much discount will you get? What will you pay for the items?

Comment

As 10% of £45 is £4.50 and 5% of £45 is £2.25, 15% of £45 is £6.75.

Alternatively, 15% of £45 = $\frac{15}{100} \times £45 = £6.75$.

The items will cost £45 − £6.75 = £38.25.

The next activity involves working out increases and decreases together.

Activity 75 Discounts [illegible]

A shop is having a sale in which everything is reduced by 10%. However, VAT also needs to be added. Does it matter whether the VAT or the discount is applied first? Try a few numerical examples first to get a feel for the problem. For example, what happens if the price of the item was £100 (before the discount and VAT have been applied)?

The [illegible] strategies [illegible] Section 3.2.

Comment

If the item costs £100, reducing the price first gives £90.

The VAT is 17.5% of £90 = 17.5 ÷ 100 × £90 = £15.75.

So the final bill is £90 + £15.75 = £105.75.

Adding the VAT first, gives £117.50.

The discount is 10% of £117.50 = 10 ÷ 100 × £117.50 = £11.75.

So the total bill is £117.50 − £11.75 = £105.75.

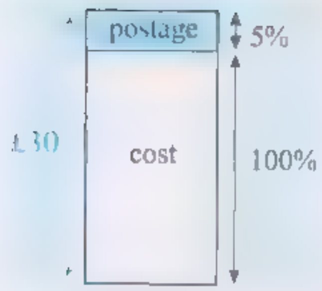
If you try some more examples, you should find that the two answers are always the same. It does not matter whether the discount or the VAT is applied first.

If the discount is 10%, the price after the discount will be 100% − 10% or 90% of the original cost. This can be found by multiplying the cost by 0.9.

If the VAT is 17.5%, the price after VAT will be 100% + 17.5% or 117.5% of the old cost. This can be found by multiplying the old cost by 1.175.

So if the discount is applied first and then the VAT, the cost will be multiplied by 0.9 and then 1.175 to give the final cost.

If the VAT is applied first and then the discount, the cost will be multiplied by 1.175, then 0.9 to give the final cost. Since it does not matter in which order the multiplications are carried out, the answer will be the same using either method.



In some situations you may be given the total after the discount or tax has been applied and asked to find the original amount. For example, suppose an online store charges 5% of the cost of the purchases for postage and packing. You have a gift voucher for £30. What is the maximum you can spend at the store, so that the total including the postage is less than £30?

From the diagram, 105% is equivalent to £30.

So 1% is equivalent to £30 ÷ 105 and 100% is equivalent to $100 \times £30 \div 105 = £28.57$ (to the nearest penny).

So the maximum amount that can be spent is £28.57. You can check your answer by working out the 5% postage charge on £28.57.

Activity 58 Working backwards

The price of an item including VAT is £18.80. What is the price before VAT is applied?

Comment

117.5% of the cost is £18.80.
So 1% is £18.80 ÷ 117.5 and 100% is $100 \times £18.80 \div 117.5 = £16$.

Thus the price before VAT was applied was £16.

Percentages are also often used to describe increases or decreases such as the rate at which prices or earnings are rising. For example, suppose last year, a bicycle cost £250 and this year the same bicycle costs £280, so it has increased in price by £30. To express this increase as a percentage of the original cost, first write it as a fraction of the original cost and then change the fraction into a percentage.

So the percentage increase is $\frac{30}{250} \times 100\% = 12\%$.

In other words, the price of the bicycle has increased by 12% over the year.

This can be summarised as follows.

$$\text{Percentage increase or decrease} = \frac{\text{actual increase or decrease}}{\text{original amount}} \times 100\%$$

Activity 59 Ups and downs

(a) The value of a car falls from £8750 to £7500 over a year. What is the percentage decrease over the year?

(b) The number of wild parrots recorded in a particular area increases from 65 to 92 over a year. What is the percentage increase over the year?

Comment

(a) The actual decrease is £8750 – £7500 = £1250.

So, the percentage decrease $= \frac{1250}{8750} \times 100\% \approx 14\%$ (to 2 s.f.).

(b) The actual increase is 92 – 65 = 27.

So, the percentage increase is $\frac{27}{65} \times 100\% \approx 42\%$ (to 2 s.f.).

Although percentages are used a lot in everyday life, they are not always understood thoroughly. In the next activity, you are asked to decide whether some statements are true or false and to explain why.

Activity 60 Is this correct?

For each of the following statements, say whether or not it is correct. Try working out some numerical examples to test your conjectures. Can you prove your results?

(a) If the price of an item is increased by 20% and then decreased by 20%, the price will remain the same.

(b) If the price of an item is decreased by 20% and I buy two of them, the total price is decreased by 40%.

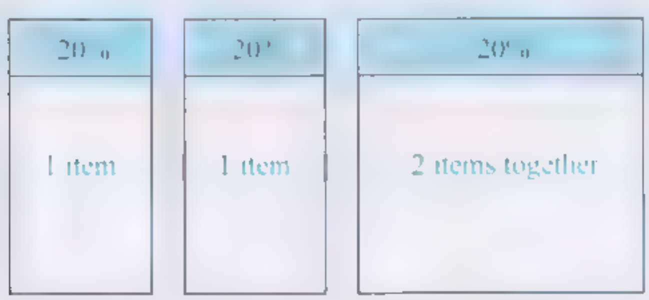
Comment

(a) Try a numerical example first, say an item costing £100. If there is a 20% increase, the new price will be £120. If we then apply a 20% decrease, we will have 80% left. 80% of £120 is $0.8 \times £120 = £96$. So the price does not remain the same.

This is true for other prices too. To apply an increase of 20%, you multiply price by 1.2 and to apply a decrease of 20%, you multiply by 0.8. This is the same as multiplying by 1.2×0.8 which is 0.96. So the new price is 96% of the original price.

(b) Suppose the item costs £100. Then with a 20% reduction, the new price is £80. So if you buy two items at £80 each, they will cost £160 instead of £200, a reduction of £40.

The overall percentage reduction is therefore $\frac{40}{200} \times 100\% = 20\%$. To help convince yourself that this is true, a diagram might help. You can see that the discount on the two items together is still only 20% of the total price.



The next activity illustrates a problem that you might already have seen in newspaper reports or elsewhere. It illustrates the importance of saying what you mean clearly.

Activity 6 Percentage points

A newspaper headline reads 'Exam pass rate increases from 50% last year to 75% this year'. Is it correct to say that the pass rate has increased by 25%? Explain why or why not.

Comment

No, the pass rate has not increased by 25%. Writing the pass rates in decimal form as 0.5 and 0.75, the percentage increase is:

$$\frac{\text{actual increase}}{\text{original}} \times 100\% = \frac{0.75 - 0.5}{0.5} \times 100\% = \frac{0.25}{0.5} \times 100\% = 50\%$$

So, the pass rate has increased by 50%.

However, when comparing percentages, the **difference** in the percentages is often described in terms of **percentage points**. So, in the last activity, you could say that there has been an increase of 75 − 50 = 25 percentage points. It is then clear that you are talking about the numerical difference between the two percentages. This is something to be aware of in media reports, particularly when interest rates are discussed. For example a report that the Bank of England has cut interest rates by 0.25% from 4.75% to 4.5% should say, strictly, that the interest rate has been cut by 0.25 percentage points to 4.5%.

Now work through Section 3.4 of Chapter 3 in the Calculator booklet.

3.4 Writing mathematics

An important part of mathematics is being able to explain your ideas, whether you are tackling an assignment, writing a report or just working on your own mathematical problem. As well as recording your thoughts and ideas, it also helps you to clarify your ideas, practise new language and notation and it encourages you to think carefully and logically. However, as with most academic writing, this is a skill that takes time and practice to develop. To understand what is involved, read through the following problem and then consider the solution below. Can you suggest any improvements?

The price of a bicycle is £250. Insuring the bike against theft and accidental damage costs £69.99 and the delivery charge is £4.99. In the sale, the price of the bicycle is reduced by 15%. How much would it cost to buy the bicycle, take out the insurance and have it delivered?

$$
\begin{aligned}
0.15 \times 250 &= 37.5 - 250 \\
&= 212.50 + 69.99 + 4.99 \\
&= \underline{287.48}
\end{aligned}
$$

First of all, can you understand the solution? You may need to jot down some notes to explain the steps. Then imagine looking back at it in a few weeks time, would you be able to understand what the problem was and how it was solved? To make any sense of it, you would need to refer back to the question and although the answer of 287.48 is numerically correct, there is no explanation of the calculations, which does make it difficult to follow – it is badly written. There are some other problems with the solution too, as indicated below.

The sale price is £250 − £37.50 so this calculation has been written down incorrectly

$$
\begin{aligned}
0.15 \times 250 &= 37.5 - 250 \\
&= 212.50 + 69.99 + 4.99 \\
&= \underline{287.48}
\end{aligned}
$$

No units have been

explanation about what this number represents.

not used correctly 0.15 × 250 is equal to 37.5 not 37.5 − 250.

So how can this solution be improved?

Explaining the different steps and writing in clear sentences will help. Enough detail needs to be included, so that whoever reads it will be able to follow the steps easily. You can often get a guide of the level of detail to include by looking at similar problems in this book or in the Exercise booklet. You can check whether you have used notation correctly and written clearly by reading it aloud.

Here is another solution to the same problem.

Write in sentences, explaining the steps

makes the presentation clear.

The reduction on the bicycle = 15% of £250
= 0.15 × £250
= £37.50.

So the sale price of the bicycle = £250 − £37.50
= £212.50.

The cost of the bicycle, insurance and delivery = £212.50 + £69.99 + £4.99
= £287.48.

Hence, the total cost is £287.48.

Use link words like so, hence, therefore to help your solution flow.

Include the units and a concluding sentence which answers the question exactly.

You may like to spend a few moments now, looking back over your own solutions. You may find that it helps to read your work out loud. Does it make sense and is it written in sentences? Are there any improvements that you can make to your own work? If there are, you might like to highlight these in the summary box below (or add them to the list) as a reminder for your future work.

Another reason for explaining your work carefully is that it will help your tutor to understand your thoughts and you may be awarded some credit for your ideas even if your final answer is incorrect.

Tips for writing mathematics

- Write in sentences, explaining your work carefully and checking that each sentence follows on logically from the previous one.
- Start each new sentence on a new line and line up the equals signs underneath each other.
- Use link words like 'hence', 'so' and 'therefore' to help your writing flow.
- Use notation correctly. Equals signs should only be used if two expressions are equal – they should not be used to link your solution together.
- Check your writing by reading it aloud – when you translate the symbols, it should still make sense.
- Use well-labelled graphs, charts and tables to summarise data and results clearly.
- Remember to include the units of measurement where appropriate.
- Include a concluding sentence which answers the question exactly.

3.5 Ratios

In Section 3.1, you saw how a fraction such as $\frac{1}{4}$ can be thought of as '1 part out of 4 parts' and also how there are often reports in the media which describe proportions in this form, for example: '9 out of 10 families with children are entitled to tax credits' or 'One in three (people) reject technology such as computers and mobile phones'. This is the same as saying '$\frac{9}{10}$ or 90% of families are entitled to tax credits' and '$\frac{1}{3}$ of people reject technology'.

Alternatively, you could say 'For every **one** person who does not use technology, there are **two** people who do.' Mathematically, we say: 'The ratio of those people who do not use technology to those who do is **one to two**.' This ratio is written as 1:2 or as the fraction $\dfrac{\text{non-users}}{\text{users}} = \dfrac{1}{2}$.

So, the number of non-users is **half** the number of users.

Many household chemicals are supplied in a concentrated form and need to be diluted before use. For example, the directions on a bottle of carpet shampoo state that 1 part of shampoo should be mixed with 9 parts water. The ratio of shampoo to water is therefore 1:9.

This ratio can also be written as a fraction:

$$\text{ratio of shampoo to water} = \frac{\text{amount of shampoo}}{\text{amount of water}} = \frac{1}{9}.$$

Suppose that there is 40 ml of shampoo left in the bottle. How much water should be mixed with it?

Since 9 times the amount of water is needed, the amount will be 9×40 ml or 360 ml.

You can also work out this from the fraction by writing down an equivalent fraction whose numerator is 40. The equivalent fraction can be obtained by multiplying the top and the bottom by 40:

$$\frac{\text{amount of shampoo}}{\text{amount of water}} = \frac{1}{9} = \frac{1 \times 40}{9 \times 40} = \frac{40}{360}.$$

So the amount of water needed is 360 ml as before.

A recipe for shortbread requires 300 g of plain flour, 100 g of caster sugar and 200 g of butter.

This means that the ratio of flour to sugar to butter is 300:100:200. Ratios can be cancelled down like fractions. So dividing by 100, this ratio can be expressed more simply as 3:1:2 or 3 parts flour, 1 part sugar and 2 parts butter.

3 parts flour 1 part sugar 2 parts butter

Now suppose you wish to make some shortbread with 300 g of butter, how much flour and sugar will you need?

Start with the butter, 2 parts is equivalent to 300 g.
So 1 part is the same as $300 \text{ g} \div 2 = 150$ g.

Hence 3 parts will be the same as $3 \times 150 \text{ g} = 450$ g.
So 450 g of flour and 150 g of sugar will be needed.

Activity 6[illegible] Using ratios

(a) A fruit drink has to be diluted by mixing 1 part of the concentrate with 5 parts of water. How much water should be added to 50 ml of concentrate? How much drink will this make?

(b) The ratio of white flour to wholemeal flour in a bread recipe is 3:7. If 150 g of white flour is used, how much wholemeal flour is needed? Can you explain why this is called a '70 per cent wholemeal' loaf?

Comment

(a) The ratio of concentrate to water is 1:5, so five times as much water as concentrate is required. If 50 ml concentrate is used, 5×50 ml or 250 ml of water is needed. If 50 ml of concentrate and 250 ml of water are used, the amount of drink will be 50 ml + 250 ml or 300 ml.

(b)

3 parts white 7 parts wholemeal

If 3 parts are equivalent to 150 g, one part is equivalent to $150\text{ g} \div 3 = 50$ g. So 7 parts are equivalent to $7 \times 50\text{ g} = 350$ g. Hence 350 g of wholemeal flour are needed.

Or you can write the ratio as a fraction and then write down the equivalent fraction whose numerator is 150:

$$\frac{\text{amount of white flour}}{\text{amount of wholemeal flour}} = \frac{3}{7} = \frac{3\times 50}{7\times 50} = \frac{150}{350}.$$

Hence 350 g of wholemeal flour is needed.

There are 10 parts altogether, and 7 parts are wholemeal, so the fraction of the flour that is wholemeal is $\frac{7}{10}$ or 70% – hence the name.

Notice how there are different ways of tackling these problems and if you prefer using a diagram to understand the situation better, that is fine.

At the beginning of this chapter you were asked to estimate how long you thought it might take you to work through it. How long did it actually take? Did some sections take more or less time than you expected? When you start the next chapter, try to estimate how long you think each section will take.

In this chapter, you have worked with fractions, decimals, percentages and ratios. Can you explain, in your own words, what these terms mean? How are the ideas the same and how are they different?

You should now be able to:

- write numbers as fractions, decimals or percentages
- do calculations with fractions, decimals and percentages
- understand how fractions, percentages and ratios are used in everyday life
- identify some of the strategies that help you to solve problems
- appreciate how to write good mathematics.

4 Patterns and formulas

Patterns occur everywhere in art, nature, science and especially mathematics. Being able to recognise, describe and use these patterns is an important skill that helps you to tackle a wide variety of different problems. This chapter explores some of these patterns ranging from ancient number patterns to the latest mathematical research. It also looks at some useful practical applications. You will see how to describe some patterns mathematically as formulas and how these can be used to solve problems both by hand and using a computer spreadsheet. At the end of the chapter, you can even have a go at an unsolved mathematical problem! Between 2000 and 2002, a \$1 000 000 prize was offered for its solution – that's just to show you that there is still a lot of very exciting mathematics to be discovered and also that everybody – you, me and all great mathematicians – do get stuck with mathematics somewhere!

You won't need a computer to do this though!

The chapter will also help you to consolidate some of the problem-solving strategies you met in Chapter 3 and it introduces some more notation to help you express ideas clearly and concisely.

4.1 Exploring patterns and processes

Suppose you are tiling a bathroom or kitchen and the first row of square tiles is to be a frieze made up of blank tiles and patterned tiles as shown below.

A friend has offered to help. How would you describe the pattern and how to arrange the tiles?

There are lots of ways of tackling this. For example, you might say that you will need some blank tiles and some patterned tiles with the 'bridges' on. Start with a bridge tile, then put a blank tile next to it. Take another bridge tile, but turn it round so that the bridge is upside down, like a smile and put it next to the blank tile in the same line. Then put another blank tile next to the smile tile. This is the pattern, just carry on: bridge, blank, smile, blank, bridge, blank, smile, … .

Or you may have decided to draw a picture of the tiles, or demonstrate the pattern with the tiles themselves. Whatever you do though, it's probably easier to remember and apply if you have recognised that the pattern is a four-tile repeat with the two different types of tile.

The frieze is an example of a type of geometrical pattern that has many applications in art, crafts and design. The next example is a number pattern which appeared in China and Persia over 700 years ago, but is still used by students in mathematical and statistical problems today and it even appears in chemistry. It is known as Pascal's triangle and the first part of it is shown below.

Blaise Pascal was a 17th century French

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

You can continue the triangle indefinitely by following the pattern.

Each row of numbers starts and ends with the number 1. Look at each pair of numbers in the last line above. For each pair, add the two numbers together and write their sum on the line below as shown. This process generates the next row of the triangle.

1 5 10 10 5 1
1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Activity 63 Pascal's triangle

You will need your triangle later in the chapter.

Using the hexagonal paper provided separately, write down the numbers in the next two rows of Pascal's triangle.

Can you spot any patterns in the numbers in the triangle?

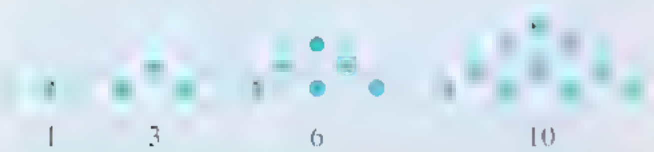
What do you notice about the sum of the numbers in each row?

If this pattern continues, what do you think the total for the 10th row will be?

Comment

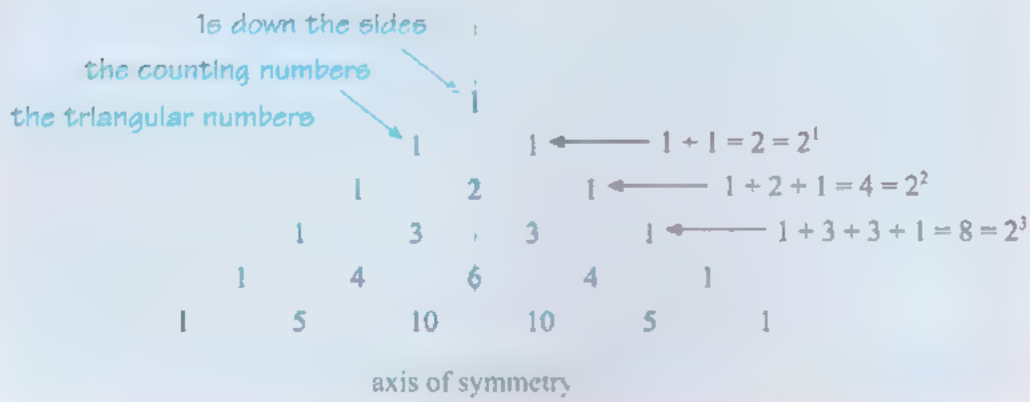
The next two rows are 1, 7, 21, 35, 35, 21, 7, 1 and 1, 8, 28, 56, 70, 56, 28, 8, 1.

There are many different patterns. The 1s down the sides of the triangle is probably the easiest one to spot! You can also see the counting numbers 1, 2, 3, 4, 5 in the diagonal rows next to the sides. In the next diagonal row, there is the sequence 1, 3, 6, 10, ... These numbers are known as the triangular numbers because they create triangular patterns.



Do not worry if you do not spot all these patterns

The triangle is symmetrical too. If you draw a vertical line down through 1, 2, 6, … , one side of the triangle is a mirror image of the other.



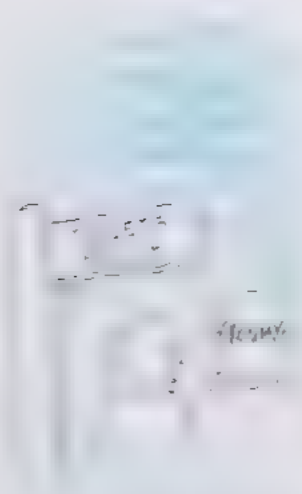
The pattern in the powers suggests that the first total, 1, might be written as 2^0. Check on your calculator to see if this is correct. Does any number to the power 0 give you 1?

Looking at the diagonal row of triangular numbers, if you add any two adjacent numbers together, you get the square numbers. For example, $1 + 3 = 4$ which is the same as 2^2; $3 + 6 = 9$ which is the same as 3^2, and so on.

Now, if you add together the numbers in each horizontal row, you get the following pattern: 1, 2, 4, 8, 16, 32, 64, … . The total for the next row is double the total for the current row. Assuming this pattern continues, the total for row 8 will be 128, row 9 will be 256 and row 10 will be 512. You might have spotted that these numbers can also be written as $1, 2^1, 2^2, 2^3, 2^4, \dots$.

What have these two examples to do with mathematics? Well, recognising patterns in shapes, sets of numbers, processes or problems generally and spotting what is the same and what is different about situations often makes the task easier to solve. You saw how recognising the tiling pattern makes it easier to remember and by using the number patterns in Pascal's triangle, you could work out the sum of each row without adding up the individual numbers. If you can spot a pattern and then describe what happens in general, this can lead to a rule or formula. Provided you can prove that this rule will always work, it can then be used elsewhere. For example, if you can work out the general process for calculating a quarterly electricity bill and then give these instructions to a computer, many electricity bills can be generated, printed and sent out in just a few minutes!

4.2 Looking for relationships



In this section, we are going to consider the relationship between quantities in two practical situations and see how to describe these relationships by writing down a general rule or a **word formula**.

Suppose you are planning a visit abroad. Your map marks all the distances between places in kilometres rather than miles. How can you work out what these distances are in miles?

The first bit of information you will need is how long one kilometre is, if it is measured in miles.

According to a dictionary, 1 km is equivalent to approximately 0.6214 miles.

The next step is to work out the mathematical process you will need to use to change kilometres into miles. You may be able to see how to convert from kilometres into miles immediately, but, if not, try to visualise a few simple examples first.

Each km is the same as 0.6214 miles as shown below.

So, if the distance is 3 km, you will have 3 lots of 0.6214 miles; if the distance is 10 km, you will have 10 lots of 0.6214 miles, if the distance is 250 km, you will have 250 lots of 0.6214 miles, and so on.

You can write this down mathematically as:

$$250 \text{ km} = 250 \times 0.6214 \text{ miles} = 155 \text{ miles (to the nearest whole number).}$$

Notice that in each of the examples above, the process for calculating the number of miles was the same: multiply the number of kilometres by the conversion rate of 0.6214. This technique will work for any distance and can be written down concisely as the following word formula:

distance in kilometres × 0.6214 = distance in miles.

This is the same as writing:

distance in miles = distance in kilometres × 0.6214.

You can use this formula to convert any distance in kilometres into miles. For example, suppose you wanted to convert 500 km. Using the formula and replacing 'distance in kilometres' by 500 gives

distance in miles $= 500 \times 0.6214 = 310.7$.

So 500 km is equivalent to approximately 311 miles.

In this example, you used the formula by replacing the phrase 'distance in kilometres' in the right-hand side of the equation by the corresponding value, 500. This is known mathematically as **substituting** the value into the formula.

Activity 64 How many miles?

How far is 350 km in miles?

Comment

Substituting 350 for 'distance in km' gives

distance in miles $= 350 \times 0.6214 = 217.49$.

So 350 km is approximately 217 miles.

Activity 65 Exchanging currencies

If you prefer, you can look up the current conversion rate in a newspaper or at a bank or travel agency.

Suppose you wish to exchange some money in pounds sterling into euros. At the time of writing, one agency had an exchange rate of €1.48 to £1 and did not make any additional charges.

(a) How many euros would you get for £5? How many euros would you get for £10?

(b) Write down a word formula that you can use to convert pounds into euros.

Your formula should start 'number of euros = …'.

(c) Check that your formula works by using it to convert £5 into euros.

Comment

(a) For each pound, you would get €1.48, as shown below.

So, if you exchange £5, you would get five lots of €1.48.

So £5 = 5 × €1.48 = €7.40.

If you exchange £10, you would get 10 lots of €1.48.

So £10 = 10 × €1.48 = €14.80.

(b) To change pounds into euros, you need to multiply the number of pounds by the exchange rate of €1.48. So the word formula is

number of euros = number of pounds × 1.48.

(c) Substituting 5 for 'number of pounds' gives

number of euros = 5 × 1.48 = 7.40.

This agrees with the answer in part (a).

What would happen [illegible] there was also a handling charge?

These two examples illustrate how a word formula can be used to summarise some mathematical process such as converting units of length or currencies. Once a formula has been derived, it can then be used in other situations, both for calculations by hand or by computer, for example for currency transactions in a bank. You will be able to practise writing down some more of your own formulas in Sections 4.4 and 4.5.

4.3 Using formulas

In the last section, we considered how a formula could be built up and then how it could be used. This section considers some more complicated formulas which have already been developed and are used in a variety of different situations – cookery, health care, business and archaeology. We hope that these examples illustrate some of the very broad applications of maths and how mathematical relationships can be used in making decisions. As you work through these examples, you may like to consider where else maths might be used in each of these topics!

When you are trying to solve a real-life problem mathematically, you often use formulas that have already been developed, so it is important to have some ideas for how to apply these.

This section will help you to do that – you may like to make your own list of tips as you work through this section and compare it with the box at the end of this section.

Maths in cookery

The time taken to cook a fresh chicken depends on its mass, as given by the following formula:

$$\text{cooking time in minutes} = 15 + \frac{\text{mass in g}}{500} \times 25$$

Roughly how long will a chicken with a mass of 2.2 kg take to cook?

To use the formula, you need to substitute the mass of the chicken into the right-hand side of the equation and then work out the resulting calculation. However, the formula asks for the mass in grams, so the first step is to convert 2.2 kg into g.

Since there are 1000 g in 1 kg, 2.2 kg = 2.2 × 1000 g = 2200 g.

Substituting 2200 for the mass in g gives:

$$\text{cooking time in minutes} = 15 + \frac{2200}{500} \times 25$$

Carrying out the multiplication and division first gives:

$$\text{cooking time in minutes} = 15 + 110 = 125.$$

So the cooking time would be about 125 minutes or 2 hours and 5 minutes.

If you prefer, you can use your calculator, but remember to make an estimate first so that you can check that your answer is reasonable.

Activity [illegible] A smaller bird

How long would it take to cook a chicken with a mass of 1.6 kg?

Comment

Converting 1.6 kg into g: 1.6 kg = 1.6 × 1000 g = 1600 g.

So the cooking time in minutes $= 15 + \frac{1600}{500} \times 25 = 95$.

The chicken would take about 95 minutes or 1 hour 35 mins.

Maths in health care

The body mass index (BMI) is sometimes used to help determine whether an adult is under- or overweight. It is calculated as follows:

$$\text{body mass index} = \frac{\text{mass in kg}}{(\text{height in m})^2}.$$

Although care needs to be taken in interpreting the results (for example, the formula isn't appropriate for children, old people or those with a very muscular physique!), a BMI of less than 20 suggests the person is underweight and a BMI of over 25 suggests the person is overweight.

In this formula, the units have been included in the expression on the right-hand side of the equals sign. It is important to change any measurements into these units before you substitute the values into the formula. (No units have been included for the BMI on the left hand side, in line with current health-care practice. However, strictly the unit for the BMI is $\frac{\text{kg}}{\text{m} \times \text{m}}$ which is the same as $\frac{\text{kg}}{\text{m}^2}$ or 'kilograms per square metre'.)

Activity 67 Body mass index

If an adult man is 176 cm tall and has a mass of 85 kg, calculate his BMI and decide whether he is overweight.

Comment

The formula needs the mass in kilograms and the height in metres. So convert the height into metres first.

Since there are 100 cm in a metre, 176 cm = 176 ÷ 100 m = 1.76 m.

Substituting the height and the mass into the formula for the BMI gives

$$\text{BMI} = \frac{85}{1.76^2} = \frac{85}{3.0976} \approx 27.$$

Since his body mass index is over 25, the man is probably overweight.

Note that, if you study science courses, scientists include the units throughout the calculation writing the solution like this:

$$\text{BMI} = \frac{85\ \text{kg}}{(1.76\ \text{m})^2} = \frac{85\ \text{kg}}{3.0976\ \text{m}^2} \approx 27\ \text{kg/m}^2.$$

Maths in business

One of the advantages of identifying the general features of a calculation and then describing it mathematically is that the formula can then be used in either a computer or a calculator program to work out many different calculations quickly and efficiently. Many utility suppliers (gas, water, electricity, telephone) have tariffs based on a fixed daily (or monthly or quarterly) charge and a further charge based on how much you have used during the billing period.

For example in 2005, a mobile phone network charged £15 per month for 30 minutes (or less) of phone calls. Extra calls above the 30 minutes were charged at 10p per minute.

Assuming that more than 30 minutes were used per month, the formula for the total monthly cost was given by

$$\text{total monthly cost in £} = 15 + (\text{total number of minutes used} - 30) \times 0.1.$$

Activity 68 Understanding the formula

Suppose that one month 75 minutes of phone calls were made. Explain how you would calculate the cost for the extra minutes (above the 30-minute allowance) and hence how to calculate the total cost for that month. Can you explain how the formula has been derived?

Comment

As 30 minutes are included in the £15 charge, the number of minutes that are charged separately is $75 - 30 = 45$.

Each extra minute costs £0.1, so 45 extra minutes will cost $45 \times$ £0.1 = £4.50.

So the total charge for that month is £15 + £4.50 = £19.50.

The formula says that the fixed charge is £15, and then each minute in excess of the 30 minutes allowed costs 10p.

This is the monthly charge of £15

The expression in the brackets works out the extra minutes that have been used

This is the cost of each extra minute in £. 10p = £0.10

total monthly cost in £ = 15 + (total number of minutes used − 30) × 0.1

Activity 69 Phone bills

On this phone tariff, what would the monthly bill have been, if the total time for the calls were the following?

(a) 48 minutes

(b) 25 minutes

Comment

(a) Substituting 48 for the total number of minutes gives:

$$\begin{aligned}\text{total monthly cost in £} &= 15 + (48 - 30) \times 0.1 \\ &= 15 + 18 \times 0.1 \\ &= 15 + 1.8 = 16.8.\end{aligned}$$

Hence the bill for the month is £16.80.

Note how this calculation is set out with a concluding sentence that answers the question precisely.

(b) The formula only applies if more than 30 minutes of calls are made, so it cannot be used in this case. This is an important step in using formulas – you do need to check that they apply in your particular situation before using them! Here, the charge is £15 for up to 30 minutes of calls, so the charge for this month is £15 overall.

Maths in archaeology

In several different parts of the world, footprints from prehistoric human civilisations have been found preserved in either sand or volcanic ash. From these tracks it is possible to measure the foot length and the length of the

stride. These measurements can be used to estimate both the height of the person who made the footprint and also whether the person was walking or running by using the following three formulas:

$$\text{height} = 7 \times \text{length of foot}$$

$$\text{relative stride length} = \frac{\text{stride length}}{\text{hip height}}$$

$$\text{hip height} = 4 \times \text{length of foot}$$

Note that no units have been included in these formulas, so it is important to make sure that the same units, for example centimetres, are used throughout the calculation. If the value of the relative stride length is less than 2, the person was probably walking and if the value is greater than 2.9, the person was probably running.

Activity 70 Footprints in the sand

From one set of footprints the length of the foot is measured as 21.8 cm and the stride length as 104.6 cm. What does the data suggest about the height and the motion of the person who made these footprints?

Comment

> [illegible] these formulas have been derived? How [illegible]

The height of the person is estimated to be 7×21.8 cm or approximately 153 cm.

To work out the relative stride length, we need to calculate the hip height and then substitute it into the relative stride length formula.

The hip height is 4×21.8 cm or 87.2 cm.

So the relative stride length $= \frac{104.6}{87.2} = 1.2$ (1 d.p.).

As the relative stride length is less than 2, the person was probably walking.

Notice that the relative stride length has no units. The stride length and the hip height are both measured in cm, and when we divide one by the other, these units cancel each other out, just like cancelling numbers. If you go on to study physics or other sciences, you will find lots of examples like this.

Notice that to find the relative stride length here, we tackled the problem by using the two formulas step by step, but you could also have combined these two formulas into one:

$$\text{relative stride length} = \frac{\text{stride length}}{4 \times \text{foot length}}$$

and used this new formula directly.

Activity 71 What's wrong?

A student used the formula: relative stride length $= \dfrac{\text{stride length}}{4 \times \text{foot length}}$ to calculate the relative stride length for the person in Activity 70. Use this

formula to work out the relative stride length and check that your answer agrees with the result in the last activity. The student entered the key sequence for the calculation as $104.6 \div 4 \times 21.8$. This gave an answer of approximately 570, so the student concluded that the person was probably running very fast indeed. Can you explain where the student made his mistake?

Comment

The calculator will perform this calculation from left to right, using the BIDMAS rules, which treat multiplication and division as equally important. So it will first divide 104.6 by 4 to get 26.15 and then multiply by 21.8 to get approximately 570. However, this is not the correct calculation from the formula. The stride length (104.6) should be divided by ($4 \times$ foot length), so the calculation should be $104.6 \div (4 \times 21.8)$ or alternatively $104.6 \div 4 \div 21.8$. Enter both these expressions into your calculator directly, and check that these both do give the correct answer.

Can you explain why these two calculations are equivalent?

Note that the student should have been expecting an answer between about 0 and 5, so an answer of 570 is suspicious! Also if the student had worked out an estimate for the answer first, the mistake might have been spotted. (If you round all the values to 1 significant figure, an estimate for the answer is $\frac{100}{4 \times 20}$ which is just over 1.)

Now, have a look back over the formulas you have used in this chapter and also any others that you might be familiar with from your home, work or hobbies. What are the common steps in using a formula? If you were asked to jot down a list of tips for using a formula for a friend, what would you say? Here are a few suggestions – feel free to add your own!

Tips for using a formula

- First, check the formula can be applied to your particular problem!
- Check what values you need to substitute and also what units these should be measured in. Convert the measurements if necessary.
- Substitute the values into the formula carefully.
- Make a rough estimate for the answer.
- Use BIDMAS to work out the resulting calculation, step by step.
- Explain your steps carefully using words like ‘substituting’ and ‘converting’.
- Check that your answer seems reasonable, both practically and from the estimate.
- Round your answer appropriately.
- Remember your concluding sentence and the units!

Although there may be many occasions when you are given a formula to use, sometimes you may need to devise your own formulas, for example if you use a spreadsheet on a computer at home or at work. This section looks at the process of devising a formula in more detail.

Part of a spreadsheet that has been constructed to record monthly income and expenditure is shown below. It is similar to a balance sheet that you might draw up by hand and includes the monthly income and outgoings, the totals and the overall balance. It is used to keep track of expenditure, particularly to try and ensure that the balance remains positive. However the spreadsheet has been stored on a computer rather than on paper and is updated regularly. One of the advantages of using a computer spreadsheet is that you can insert formulas into the spreadsheet to carry out the necessary calculations automatically, without using your calculator.

A spreadsheet is made up of rows and columns of cells. The columns are identified by letters and the rows by numbers. This enables you to identify each cell in the spreadsheet. For example the number 110.56 is in column B, row 11, so this cell is B11.

column B

	A	B
1		Amount/£
2	**Monthly Income**	
3	Salary	850.28
4	Other Income	25.67
5	**Total Income**	875.95
6		
7	**Monthly Outgoings**	
8	Rent	409.91
9	Food	116.57
10	Transport	87.20
11	Regular Bills	110.56
12	Other	68.35
13	**Total Outgoings**	792.59
14		
15	**Balance**	83.36

row 11

cell B11

Cells can be found from their reference, by looking down the column and across the row. So cell A3 can be found by looking down column A and across row 3. This cell contains the word 'Salary'. Notice that cells can contain either text or numbers.

Activity 72 Spreadsheet cells

(a) What is contained in the following cells?

(i) A5 (ii) A12 (iii) B15 (iv) B1

(b) What is the reference for the cells that contain the following?

(i) The number 792.59

(ii) The word 'Food'

Comment

(a) (i) The cell that is in both column A and row 5 contains the words 'Total Income'.

(ii) Looking down column A and across row 12, cell A12 contains the word 'Other'.

(iii) The cell that is in column B, row 15 contains the number 83.36.

(iv) The cell that is in column B, row 1 contains the heading 'Amount /£'. This indicates that the entries in this column are amounts of money and that they are measured in £. This heading could also have been written as 'Amount in £' or 'Amount (£)'.

(b) (i) 792.59 is in column B and row 13, so its reference is B13.

(ii) 'Food' is in column A and row 9, so its reference is A9.

Now have another look at the spreadsheet and see if you can work out what information it shows. For example, if you look at row 3, this shows that the monthly salary is £850.28. Although cell B3 only contains the number 850.28, you know that this is measured in pounds from the heading in cell B1. Overall, the spreadsheet shows the different items that make up the monthly income and where money has been spent over the month as well. If you were keeping these records by hand, you would then need to calculate the total income, the total outgoings and the balance.

For example, to find the total income for the month, you would need to add the salary (£850.28) to the other income (£25.67). This gives the total income of £875.95. In other words, to calculate the value in cell B5, you need to add the values in cells B3 and B4 together. This can be written as the following formula:

value in B5 = value in B3 + value in B4

Activity 73 Finding the balance

Formulas have also been used to calculate the total monthly outgoings and the balance. If you were working these calculations out by hand, what would you do? Check by comparing your answers with the values in cell B13 and B15.

Try to write down the formulas for these cells in the form 'value in B13 = …'

Comment

To calculate the total monthly outgoings, you need to add up the individual outgoings on 'Rent', 'Food', 'Transport', 'Regular Bills' and 'Other'. So the formula will be:

value in B13 = value in B8 + value in B9 + value in B10 + value in B11 + value in B12

To find the balance, you need to take the outgoings away from the total income. So the formula will be:

value in B15 = value in B5 – value in B13

To put these formulas in the spreadsheet, you can type the formula directly into the relevant cell, starting with an equals sign to show that you are entering a formula rather than a word.

	A	B
1		Amount/£
2	**Monthly Income**	
3	Salary	850 28
4	Other Income	25 67
5	**Total Income**	=B3+B4
6		
7	**Monthly Outgoings**	
8	Rent	409 91
9	Food	116 57
10	Transport	87 2
11	Regular Bills	110 56
12	Other	68 35
13	**Total Outgoings**	=SUM(B8 B12)
14		
15	**Balance**	=B5 - B13

Notice that in cell B13, a shorthand form for the sum has been entered. You could have typed in = B8+B9+B10+B11+B12, but it is also acceptable to use the shorthand form, = SUM(B8:B12), which is shown here. This instruction tells the computer to add together the values in all the cells from B8 to B12.

One of the advantages of using a spreadsheet is that if you change some of the numbers, all the calculations that use that particular number will be automatically updated to reflect the change. For example, in this budget the amounts for the salary, rent and the regular bills are likely to remain the same from one month to the next and may only be updated once or twice a year. However, the amounts for food, transport, other bills and other income probably will change from month to month. These values can be changed easily on the spreadsheet and the revised balance produced immediately.

Activity [illegible] Spreadsheet formulas

The formulas used in a spreadsheet are shown in the diagram below. Note that * is the notation for multiplication and / is the notation for division.

	A	B	C	D
1	Item	Cost/£		Total/ £
2	Radio	64.99	=B2*0.175	=B2+C2
3	Kettle	16.49	=B3*0.175	=B3+C3
4	Fan	12.99	=B4*0.175	=B4+C4
5			Total	

(a) The values in columns C and D are displayed to two decimal places as they represent an amount of money. What values will be displayed in cell C3 and cell D3?

(b) What do you think is being calculated in the cells in column C? Can you suggest a suitable heading for this column, to be entered into C1?

(c) Cell D5 should contain the total of the values in D2, D3 and D4. Write down a formula that could be entered in cell D5. What does this value represent?

Comment

(a) The formula in C3 is:

value in C3 = value in B3 × 0.175.

So the value in C3 = 16.49 × 0.175 = 2.885 75.

Although the full value is kept in the cell, it will only be displayed to 2 d.p. as 2.89.

The value in D3 is obtained by adding together the values in B3 and C3.

So, value in D3 = value in B3 + value in C3 = 16.49 + 2.885 75 which is 19.375 75. This will be displayed to 2 d.p. as 19.38.

(b) The numbers in column C are 0.175 or 17.5% of the values in column B. Thinking back to Chapter 3, 17.5 % is the main UK VAT rate, so it looks as if column C represents the amount of VAT paid on the different items described in column A. A suitable heading might be VAT/£ or VAT in £ or VAT (£).

(c) The total can be found by adding together the values in cells D2, D3 and D4. So the formula '= D2+D3+D4' could be entered into cell D5. Alternatively, you could use '=SUM(D2:D4)'. This represents the total cost (including VAT) of the radio, kettle and fan.

With these changes, the spreadsheet will look like the following.

	A	B	C	D
1	**Item**	**Cost/£**	**VAT/£**	**Total/ £**
2	Radio	64 99	11 37	76 36
3	Kettle	16.49	2 89	19 38
4	Fan	12 99	2 27	15 26
5			**Total**	111 00

4.5 Your own formulas

In the last section, you worked out some formulas that could be used in a spreadsheet. This section gives you some more practice in deriving formulas both by looking at some number tricks and rearranging some existing formulas.

Activity 75 Think of a number

Before you start, cover up the comment below! Now try this number puzzle. (If you use your calculator, remember to press the equals sign after each instruction.)

'Think of a number. Add 5. Double it. Subtract 8. Divide by 2. Take away the number you first thought of. Add 4.'

Now if 1 represents the letter 'A'; 2 represents the letter 'B'; 3 represents the letter 'C'; and so on, work out the letter represented by your answer and write down the name of an animal beginning with this letter.

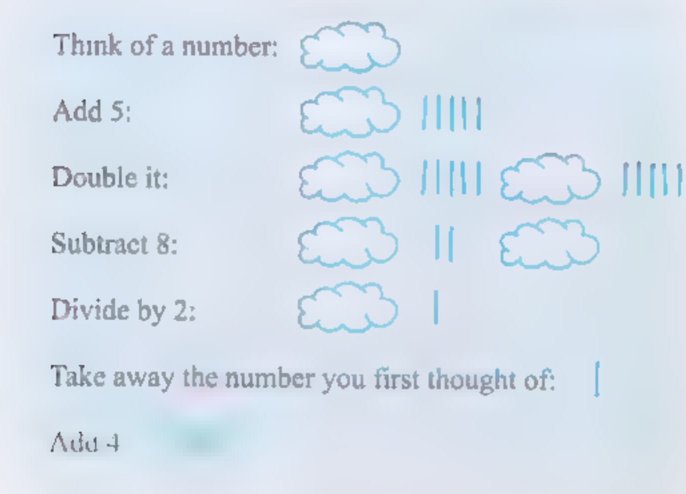
Comment

If the number 3 were chosen initially, the instructions work out as follows.

Add 5:	$3 + 5 = 8$
Double it:	$8 \times 2 = 16$
Subtract 8:	$16 - 8 = 8$
Divide by 2:	$8 \div 2 = 4$
Take away number:	$4 - 3 = 1$
Add 4:	$1 + 4 = 5$

You will find that you always get 5, whatever number you start with. This gives the letter E. Did you choose an elephant?

To see how this trick works, read through the instructions below. Because we do not know what number you have thought of, we have replaced the number by a thought bubble like this:



This shows that the numerical answer is always five – it does not depend on which number was chosen first of all. So the letter of the alphabet chosen is E. There are not many animals with names beginning with E and most people do think of an elephant first, but you might be unlucky if someone chooses an emu or an eel!

Rather than using a cloud to represent the number and explaining the trick pictorially, you could write down expressions for the answer at each stage and use either a word (for example, number) or a letter (for example, N) to represent the number. The trick can then be written as follows.

Instruction	**Expression**
Think of a number	n
Add 5	$n + 5$
Double it	$n + 5 + n + 5 = 2 \times n + 10$
Subtract 8	$2 \times n + 10 - 8 = 2 \times n + 2$
Divide by 2	$(2 \times n \div 2) + (2 \div 2) = n + 1$
Take away the number you first thought of	$n + 1 - n = 1$
Add 4	$1 + 4 = 5$

Try making up your own number tricks. What makes a good trick?

Try the following trick several times. Think of a number between 1 and 10. This will work with numbers greater than 10, but the restriction is to keep the arithmetic manageable. Multiply by 4. Add 6. Divide by 2. Subtract 3. Divide by 2, and your answer is?

Write down the number you first thought of and your answer. What do you notice? Can you explain why this happens, either by using a diagram or by writing down the expressions for the answer at each stage?

Comment

You should find that this time the answer is always the number you chose at the start. The expressions for the answer at each stage are shown below.

Instruction	Expression
Think of a number between 1 and 10	n
Multiply by 4	$4 \times n$
Add 6	$4 \times n + 6$
Divide by 2	$(4 \times n \div 2) + (6 \div 2) = 2 \times n + 3$
Subtract 3	$2 \times n + 3 - 3 = 2 \times n$
Divide by 2	$2 \times n \div 2 = n$

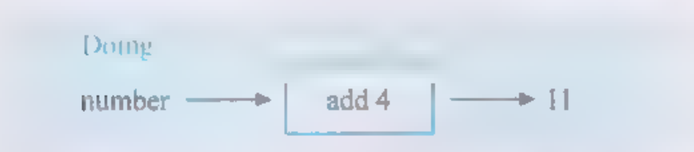
Now try the following.

Think of a number. Add 4. If my answer is 11, can you work out what number I was thinking of?

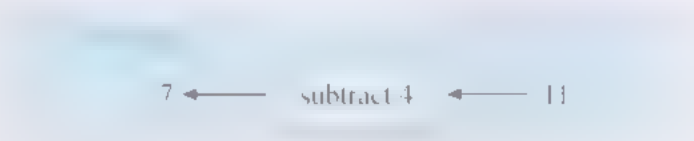
You might have said 'What number do I have to add on to 4 to get 11?' or perhaps 'If I take away 4 from 11 what number do I get?' In both cases you should have arrived at the answer 7.

In the second method 'subtracting 4' undoes the 'adding 4' in the original instructions.

This process can be illustrated by a 'doing–undoing diagram'.



In the doing part of the diagram, start with the number and write down the operations applied in turn until you get the answer. Here there is just one operation 'add 4'.



For the undoing part of the diagram, start on the **right** with the answer, in this case 11. Then work back towards the **left**, undoing each operation in turn until you find the starting number. In this case, 'subtract 4' undoes 'add 4' and $11 - 4 = 7$. So, the number first thought of was 7. Notice how the arrows indicate the direction to read the diagram.

The next activity gives you some practice with doing and undoing. You will find these techniques helpful for rearranging formulas later. Part (d) involves two stages.

Activity [illegible]: Doing and undoing

Try to work out what number I was thinking of in the following problems. You may find it helpful to use some doing–undoing diagrams.

(a) Think of a number, subtract 3 and the answer is 2.

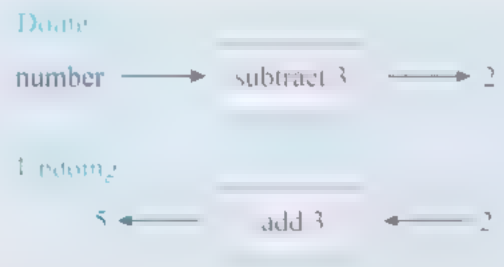
(b) Think of a number, multiply by 5 and the answer is 35.

(c) Think of a positive number, square it and the answer is 81.

(d) Think of a number, add 4, double it and the answer is 14.

Comment

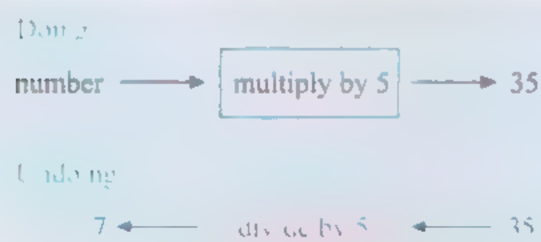
(a)



So the number I thought of was 5.

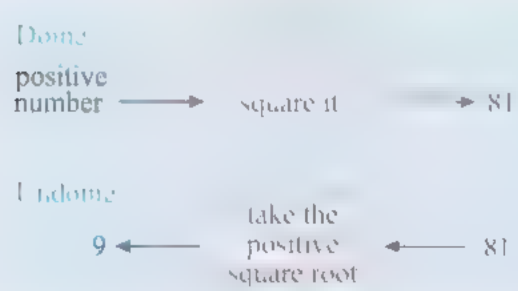
Remember you can check your answer by using it in the trick: $5 - 3 = 2$.✓

(b)



So the number I thought of was 7. (Check: $7 \times 5 = 35$. ✓)

(c)



So the number I thought of was 9. (Check: $9^2 = 81$. ✓)

(d)

Doing

number $\longrightarrow$ $\boxed{\text{add 4}}$ $\longrightarrow$ $\boxed{\text{multiply by 2}}$ $\longrightarrow$ 14

Undoing

3 $\longleftarrow$ $\boxed{\text{subtract 4}}$ $\xleftarrow{7}$ $\boxed{\text{divide by 2}}$ $\longleftarrow$ 14

So, working from the right, $14 \div 2 = 7$ and $7 - 4 = 3$.

If you like, you can write the output from each box on the arrow as shown above.

Hence, the number I thought of was 3. (Check: $3 + 4 = 7$; $7 \times 2 = 14$.✓)

When you have to deal with more than one operation, just take each step in turn. Write down the doing diagram and draw the undoing one underneath, working backwards to undo each operation.

To summarise, the following operations undo each other:

- addition and subtraction
- multiplication and division
- squaring and taking the positive square root.

Activity [illegible]

Think of a number. Add 5. Multiply by 3. Subtract 4.

If the answer is 17, can you work out what number I was thinking of?

Here there are several steps, so to find the number it will be necessary to undo each of these steps in turn, starting with the last step. Draw the doing diagram and then the undoing one beneath it.

Comment

The diagram below shows the steps taken in this case and then how to undo each one.

Doing

number $\longrightarrow$ $\boxed{\text{add 5}}$ $\longrightarrow$ $\boxed{\text{multiply by 3}}$ $\longrightarrow$ $\boxed{\text{subtract 4}}$ $\longrightarrow$ 17

Undoing

2 $\longleftarrow$ $\boxed{\text{subtract 5}}$ $\xleftarrow{7}$ $\boxed{\text{divide by 3}}$ $\xleftarrow{21}$ $\boxed{\text{add 4}}$ $\longleftarrow$ 17

So if the answer is 17, undoing the steps by working from right to left gives:

add 4: $17 + 4 = 21$

divide by 3: $21 \div 3 = 7$

subtract 5: $7 - 5 = 2$

So the number I thought of was 2. (Check: $2 + 5 = 7$; $7 \times 3 = 21$; $21 - 4 = 17$. ✓)

The same sort of technique can be used to change formulas around. For example, earlier in the chapter, you found a formula to convert pounds into euros:

number of euros = number of pounds × 1.48.

However, what if you wished to convert euros into pounds, say while you were shopping on holiday?

Then you would need a formula for the number of pounds based on the number of euros. We can tackle this by drawing the 'doing and undoing' diagrams for this situation:

Doing

number of pounds ⟶ | multiply by 1.48 | ⟶ number of euros

Undoing

number of pounds ⟵ | divide by 1.48 | ⟵ number of euros

So, starting at the right of the undoing diagram, the formula for converting euros into pounds is:

number of pounds = number of euros ÷ 1.48.

Activity 79 Miles and kilometres

To change kilometres into miles, you used the formula:

distance in miles = distance in kilometres × 0.6214.

Starting with 'distance in kilometres', draw a doing diagram to get the distance in miles. Draw the undoing diagram and hence write down the formula for changing miles into kilometres.

Your formula should start 'distance in kilometres ='.

Comment

The doing and undoing diagrams are shown below.

Doing

distance in km → multiply by 0.6214 → distance in miles

Undoing

distance in km ← divide by 0.6214 ← distance in miles

So the formula for converting miles into kilometres is

distance in kilometres = distance in miles ÷ 0.6214.

Activity 30 How many minutes for chatting?

Recall the formula for the monthly cost in £ of a mobile phone that we used earlier:

monthly cost in £ = 15 + (total number of minutes − 30) × 0.1.

The owner wishes to stick to a monthly budget of £25. Starting with the 'total number of minutes', draw a doing diagram to show the operations to find the 'monthly cost'. Put the monthly cost equal to £25 and then draw an undoing diagram to work out how many minutes of phone calls can be made if the monthly cost is £25.

Comment

Here is the doing diagram.

Doing

number of minutes → subtract 30 → multiply by 0.1 → add 15 → 25

Undoing each step in turn gives the following undoing diagram.

Undoing

130 ← add 30 $\xleftarrow{100}$ divide by 0.1 $\xleftarrow{10}$ subtract 15 ← 25

So applying these operations in turn:

subtract 15 gives 25 − 15 = 10,

divide by 0.1 gives 10 ÷ 0.1 = 100,

add 30 gives 100 + 30 = 130.

Hence, 130 minutes can be used for a monthly budget of £25.

With practice, this process of using a doing–undoing diagram does become second nature. In some cases, it can be useful if you need a formula in a different form. However, for complicated formulas, a different approach (which you will meet if you continue your mathematical studies) is often used.

4.6 Proportion

In this section, we consider two important types of relationship that occur frequently in real life: direct and inverse proportion.

Direct proportion

The first type of relationship is known as **direct proportion**. Two quantities are said to be directly proportional to each other if when one doubles, triples, quadruples, the other also doubles, triples, quadruples. For example, if you buy three times as many items as usual, you would expect to have to pay three times as much money (unless there was some special offer available), because the price is directly proportional to the number of items bought.

Activity [illegible] Directly proportional relationships

Which of the following quantities are in direct proportion?

(a) The number of kilometres and the equivalent number of miles in Activity 79.

(b) The number of euros that are exchanged for pounds in Activity 65.

(c) The monthly cost of a mobile phone and the minutes used for talking in Activities 68 and 69.

Comment

Parts (a) and (b) are both examples of direct proportion. If you multiply the number of kilometres by any factor, the number of miles will also change by this factor. For example, if you double the number of kilometres, the number of equivalent miles will also double, and similarly for exchanging currency. If you triple the number of pounds you exchange, you would expect to get three times as many euros. However part (c) is not a directly proportional relationship. If you use 30 minutes, the charge is £15. But if you double the time used to 60 minutes, the charge is £18, and the price has not doubled.

You can tackle problems involving direct proportion in many different ways – using a formula is not always the easiest way, as the following example shows.

Suppose that in a tea club at work, a group of people share a carton of milk each day, but provide their own tea. The carton contains enough milk for 12 cups of tea. If the carton costs 36p, how much should someone who has four cups of tea with milk pay?

Note how the problem has been simplified by considering just one cup of tea first!

This person had $\frac{4}{12} = \frac{1}{3}$ of the milk, so should pay $\frac{1}{3}$ of the cost of the milk which is 12p.

Alternatively, you can work out the cost of milk for one cup of tea as a first step.

Since there is enough milk for 12 cups, one cup will cost $36\text{p} \div 12 = 3\text{p}$.

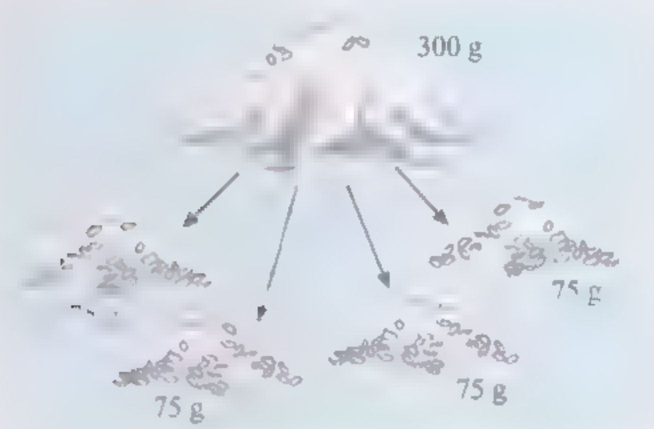
So four cups will cost $4 \times 3\text{p}$ or 12p.

Activity [illegible] [illegible]

A recipe for a vegetable curry for four people requires 300 g of rice. By first working out how much rice one person requires, calculate how much rice would be needed for nine people.

Comment

Work out the amount of rice for one person: $300\text{ g} \div 4 = 75\text{ g}$.



So nine people will need $9 \times 75\text{ g} = 675\text{ g}$.

(You can work this out in other ways too!)

Activity [illegible] [illegible]

A patient needs to have 600 mg of a drug each day, given in three doses. Each tablet contains 100 mg of the drug. How many tablets are needed for each dose?

Comment

One tablet contains 100 mg of the drug.

So six tablets will contain 600 mg of the drug. The patient will need to take six tablets spread over three doses.

So the number of tablets in each dose will be $6 \div 3 = 2$.

Remember to check that your answer seems reasonable!

You can also use a formula here:

$$\text{total number of tablets} = \frac{\text{amount of drug required}}{\text{amount of drug in one tablet}}.$$

Use the formula to see if you get the same answer.

Inverse proportion

The second type of relationship is known as **inverse proportion**.

Suppose you have decided to hire a taxi to take a group of colleagues from work to the railway station. If the taxi firm charges a **set fee for the journey**, then the more people who go in the taxi, the less each person has to pay. If two people go, each pays half the cost; if three people go, each pays a third of the cost and if four people go, each person pays a quarter of the cost. This is an example of inverse proportion. As one quantity (the number of people) doubles or triples, the other quantity (the cost) halves or is reduced to a third of the original value. In other words if you multiply one quantity by a factor, the other quantity is divided by that factor.

The cost per person can be calculated directly or by using the formula:

cost per person in £ = taxi fare in £ ÷ number of people.

For example, if the fare is £7.80 and four people go in the taxi, each person will have to pay £7.80 ÷ 4 = £1.95.

Two quantities are said to be inversely proportional if their product is a constant. For example, suppose you go on a journey of 30 km and it takes you an hour to get there. Your average speed for the journey is 30 km per hour. Now if you doubled your speed to 60 km per hour, it would take you half as long (that is, half an hour) to get there; if you halved your speed, it would take twice as long and so on. In each case, the following relationship holds:

(speed in km per hour) × (time for journey in hours) = (distance travelled in km)

This shows that for a given distance speed and time are inversely proportional to each other.

This formula can also be written as:

$$\text{speed in km per hour} = \frac{\text{distance travelled in km}}{\text{time for journey in hours}}.$$

Activity 8[illegible] How fast?

For the journey of 30 km above, what is the average speed if the journey took $1\frac{1}{2}$ hours?

Comment

Substituting the distance as 30 and the time as the decimal 1.5 in the formula above gives:

$$\text{average speed in km per hour} = \frac{30}{1.5} = 20.$$

So the average speed is 20 km per hour.

4.7 Inequalities

In this chapter, there have been three occasions when checks have been made to see whether the result is greater than or less than some other value. The first case was in calculating the BMI and determining whether the person was overweight or underweight, the second case was in determining whether a person was walking or running from his footprints; and the third case was in checking whether a phone had been used for more than 30 minutes. Checking whether values are greater than or less than some limit happens frequently in everyday problems, particularly in safety limits but elsewhere too. For example, medicines may have to be stored at a temperature of 25 °C or less. Child rail tickets can be bought for children who are 5 or more years old but less than 16 years old.

Rather than writing out 'greater than' or 'less than', some shorthand notation is often used as shown below.

If you have difficulty interpreting these symbols, you can think of them as arrows which point to the smaller number.

Inequalities

$>$ greater than

$\geq$ greater than or equal to

$<$ less than

$\leq$ less than or equal to

The symbols can be read from left to right. For example, $11 > 9$ is read as '11 is greater than 9' and holiday cost (in £) < 1000 is read as 'holiday cost is less than £1000'.

To use the symbols in your own writing, decide what you want to say first, then use the symbol. For example, since 10 is greater than 5, this could be written as $10 > 5$ or since on the number line $^{-}4$ lies to the left of $^{-}3$, $^{-}4$ is less than $^{-}3$ and this could be written as $^{-}4 < {}^{-}3$.

Similarly, the instructions for the medicine could be written as:

medicine temperature in °C ≤ 25

Sometimes, you may find it helpful to draw a number line to visualise this sort of information. For example, the ages which children can claim the child rail fare are from their fifth birthday up to but not including their sixteenth birthday. This means that their age has to be greater than or equal to 5 and less than 16.

Note how drawing a diagram here helps you to see what is happening.

The empty circle means that this number (16) is not included and the filled-in circle means that this number (5) is included in the interval. This can be written as:

$5 \leq$ age for child rail fare < 16.

(a) Put the correct symbol (< or >) in each of the boxes below.

(i) 4 □ 7

(ii) 18 □ 10

(iii) 3 □ $^{-}2$

(b) Explain what the following mathematical statements mean.

(i) Balance in account > 0.

(ii) Speed (in mph) on motorway ≤ 70.

(iii) $18 \leq$ age (in years) ≤ 50.

(c) Express the following conditions using the inequality symbols.

(i) The fridge temperature should be less than 4 °C.

(ii) There must be at least five people on the committee.

(iii) For an ideal weight, a person's BMI should be greater than 20 and less than 25.

Comment

(a) (i) $4 < 7$

(ii) $18 > 10$

(iii) $3 > {}^{-}2$

(b) (i) The balance in the account is greater than zero, or in credit.

(ii) The speed on the motorway is less than or equal to 70 mph.

(iii) The age is between 18 and 50 years inclusive.

(c) (i) Fridge temperature (in °C) < 4.

(ii) Number of people on committee ≥ 5.

(iii) $20 < \text{BMI} < 25$.

Inequalities are also used a lot in computer programs to check whether conditions have been fulfilled. For example, if the balance in your bank account is negative, you may be prevented from withdrawing money from a cash machine.

[illegible] the time of writing.

A lot of the mathematics you have looked at so far in the course has been used to help people solve problems for centuries. However there is much more to mathematics than that. Exciting and new developments are being made all the time and there are many problems that mathematicians have not managed to work out. This section takes a brief look at one of these problems.

A conjecture is a suggestion that has not yet been proved to be true.

In 1742, a Prussian mathematician, Christian Goldbach, made the following conjecture:

> All positive even integers bigger than or equal to 4 can be expressed as the sum of two primes.

This statement contains quite a lot of mathematical terminology, so before going further, here are some definitions.

- An **integer** is a whole number, negative, zero or positive; for example, 6, 23, 0 and 281 are all integers.
- An **even** integer is one that can be divided exactly by two; for example, 8, 0, 42, 128 or 1002.

  The last digit of an even integer is always 0, 2, 4, 6 or 8. Integers that are not even are **odd**.

A lot of encryption processes depend on prime numbers.

- A **prime** number is one that can only be divided exactly by 1 and itself. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59.

Now, break down Goldbach's conjecture into steps.

'All positive even integers bigger than or equal to 4 …' means the numbers 4, 6, 8, 10, 12, and so on.

'… can be expressed as the sum of two primes' means that you can choose any of the numbers from the prime number list to add together. For example, $8 = 5 + 3$.

Activity 56 Prime numbers

Try to express the following numbers as a sum of two primes.

4 12 28 40 62

For each number, try to find as many combinations of two primes as you can.

Do you think Goldbach's conjecture is true?

Comment

It can help if you look at the list of prime numbers systematically, starting with 2. Is there a number in the list that you can add on to 2 to get the number you want? Then look at 3, and so on. You should be able to find the following combinations:

$4 = 2 + 2$;

$12 = 5 + 7$;

$28 = 5 + 23$ or $28 = 11 + 17$;

$40 = 3 + 37$ or $40 = 11 + 29$ or $40 = 17 + 23$;

$62 = 3 + 59$ or $62 = 19 + 43$ or $62 = 31 + 31$.

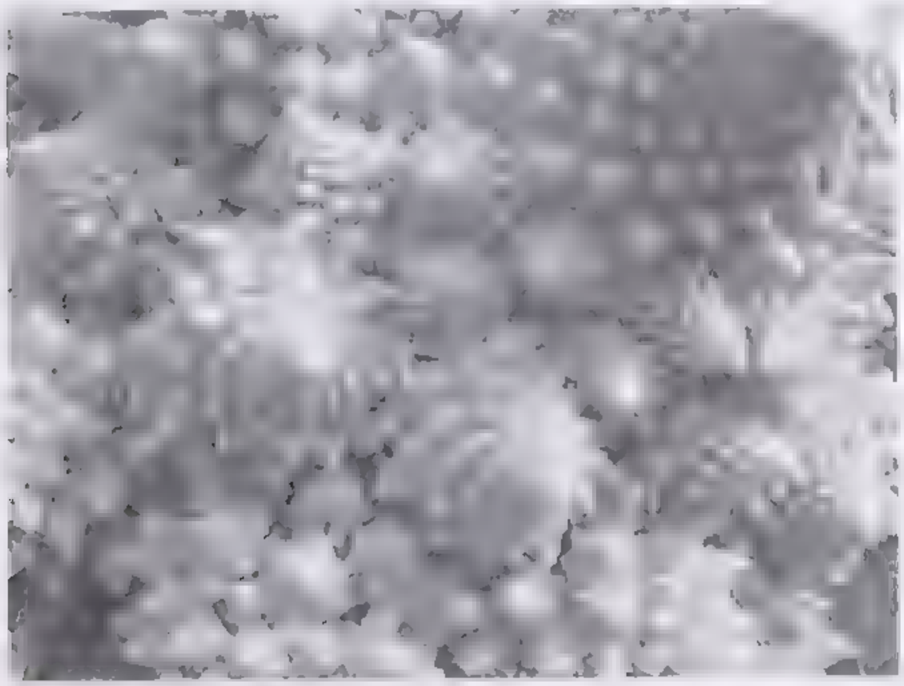
Are you convinced that Goldbach's conjecture is always true?

There is also a $1 000 000 prize on offer for each of the seven great unsolved problems of mathematics known as the Millennium Problems.

Although the conjecture does work for the few numbers we have tried here, that does not necessarily mean that it is going to work for all the even integers greater than or equal to four. The even integers go on for ever, so it is not possible to check them all individually either by hand or computer – some other way of proving the result is needed. Although higher branches of mathematics have been successful in tackling a lot of similar problems, this one is turning out to be particularly tough. So tough in fact that, in March 2000, the publishers Faber and Faber offered a $1 000 000 prize to anyone who managed to prove the conjecture by March 2002 and no one did! So even though many great mathematicians have attempted a proof, over 250 years later, it still hasn't been sorted out. You might find that quite reassuring the next time you get stuck on a mathematics problem – being stuck really is a natural state for a mathematician! It's often the time when great discoveries are made or when the best learning takes place.

4.9 Patterns in nature and elsewhere

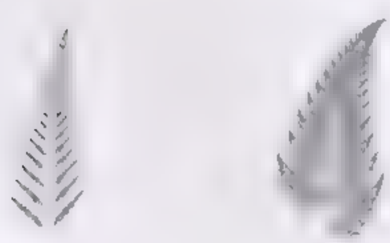
If you take a cauliflower and break off one of the florets, the floret appears to be just the same shape as the original cauliflower but on a much smaller scale



Romanescu cauliflower

This idea of self-similarity occurs throughout nature – the frond of a fern looks like the whole fern, a branch of a tree splits into further branches which look like trees themselves, and so on.

Self-similarity now forms part of an area of mathematical research known as fractal geometry. Beautiful patterns can be generated by fractal formulas, such as the Barnsley fern and the fractal below. Fractal geometry also has many practical applications in the physical sciences, medical research, economics and computing, particularly in compressing images.



Barnsley ferns

Fractal

You can generate part of a fractal yourself, by using Pascal's triangle that you constructed earlier in the chapter. If you shade in (or cover) all the odd numbers in the triangle, a pattern starts to emerge as shown below.

Wacław Sierpinski (1882–1969) was a Polish mathematician.

This is part of a fractal known as Sierpinski's triangle. It can also be generated by starting with a triangle, splitting it into four smaller triangles and removing the middle one. Then split each of the remaining triangles into four smaller triangles and remove the middle one. Carrying on in this way results in something like the sequence below.

Constructing Sierpinski's triangle

4.10 Summing up

In this chapter, you have looked at a variety of problems all of which involved using patterns or formulas and you have also extended some of your strategies for solving problems. One of the first steps in tackling any problem is to check that you understand both the problem and the information you have been given. In Chapter 3, this step concentrated on what the question meant. However, this can also involve looking up or checking on mathematical terms, notation or definitions as in Goldbach's conjecture. You may be able to sort this out yourself by referring back to your notes or books or, if it is an unfamiliar situation, it may be easier to discuss it with someone else. Understanding the notation is also important. The notation used for inequalities is often used to express ideas concisely and you may find it helpful to add this to your dictionary or to your notes for future use.

You have already seen how diagrams can be helpful and that applies to this chapter as well (for example a number line when dealing with inequalities).

Visualising a problem in a practical way, as in the currency exchange, can help you to work out what to do next as well. You may also like to try working through a problem with a few simple numbers first to get a feel of what's involved. This strategy was used in Goldbach's conjecture. However although we broke Goldbach's conjecture down into steps and found that the conjecture was true for all the examples we tried, this does not prove that the conjecture is **always** true. A pattern isn't proof!

Using formulas is an important skill, but it is important to check that the formula is appropriate for your particular problem (see the mobile phone problem) and also that any measurements are in the same units as those specified in the formula. It is easy to make mistakes in this way, so checking that the answer you obtain seems reasonable is very important too. You may like to make an estimate first as well. It is also important to check formulas you have derived yourself, particularly if you are using a spreadsheet. If you work out the numerical answer to the calculation first, you can then check that you get the same result when you use the formula in the spreadsheet. Although this doesn't guarantee that you have got the right formula (you may make the same mistake twice after all!), it can help to eliminate careless mistakes.

Activity [illegible] How are you getting on?

Spend a few minutes reviewing your work on this chapter using the study checklist below. Are there any topics that you feel you need further practice on or that you would like to discuss with your tutor?

Study checklist

You should now be able to:

- visualise problems using pictures and diagrams
- recognise patterns in a variety of different situations
- use a word formula to help solve a problem
- derive simple word formulas of your own, for example for use in a spreadsheet
- use doing and undoing diagrams to change formulas round
- solve problems involving direct and inverse proportion
- interpret and use notation for inequalities
- appreciate that there are many unsolved problems in mathematics.

5 Shapes and sizes

This chapter is about shapes and considers how some of their properties (such as area or volume) can be measured. It is a practical chapter, so you will need some basic equipment to tackle the activities – a 30 cm ruler, some scissors, glue, a piece of thin card (a cereal packet or similar will do) and a sheet of graph paper.

The word geometry comes from two Greek words – 'geo' means earth and 'metros' means measurement.

You met measurement in Chapter 2 where the SI system was introduced. Section 1 extends those ideas by considering measuring very small things (viruses, bacteria or germs) and very large things (galaxies, distances in the universe) and shows how these measurements can be written down. It also explains some important mathematical techniques in working with powers, which will be useful in your further mathematical or scientific studies, and how using an appropriate notation can make it easier to carry out calculations and understand problems.

One of the difficulties in considering very large or very small things is in visualising them. You have already seen how being able to draw a diagram which represents a situation helps you to visualise the problem; everyday examples of this are maps which represent an area of the country, plans of a room or a garden, and scale drawings of a model or component. Section 2 explores these ideas in more depth and builds on the ideas on ratio in Chapter 3. You will be asked to use a scale diagram to produce your own model of a Chinese puzzle, which you will need for other activities in the chapter.

Section 3 introduces some basic properties of shapes such as triangles, rectangles and circles and also introduces some new vocabulary. Section 4 introduces perimeters of various shapes including circles and uses this knowledge to solve a practical problem.

As well as measuring lengths, many practical situations involve measuring areas: how much paint do I need for the wall? What is the floor area in that office? Section 5 shows how some formulas can be used to calculate areas and how these formulas can be used in practical situations.

Section 6 extends these ideas to volumes: how much will that box hold? How much water do I need for the fish tank? What size radiators will I need for the room? Throughout Sections 4, 5 and 6, the relationship between different units is stressed.

You will use your calculator to explore scientific notation and to practise dealing with very large and very small numbers. You will also have the chance to calculate roots and to work out a variety of practical problems.

The study skills in this chapter concentrate on understanding and using new notation and vocabulary and extending your problem-solving strategies further. One of the problem-solving strategies you have already met is drawing a diagram to help you to visualise a problem and this chapter looks at the use of diagrams and models in more detail. You will also meet some

more abstract mathematical problems as well. Problems can be tackled in a variety of different ways and deciding which approach is most appropriate in a particular situation is important. To illustrate this, in Section 5 we consider which mathematical technique is best in a medical emergency!

5.1 Big and small

In Chapter 2, you used the SI system for some everyday measurements and considered problems, such as arranging kitchen units, giving doses of medicine and even looking at the components of vitamin pills. However you may have to deal with much larger and much smaller quantities than these, particularly if you are interested in science or technology. Just for a moment, think of the biggest and smallest physical objects or numbers that you can and consider how you would describe their size to someone else.

You may have thought of the width of the universe, the distance to a star or the number of grains of sand in the world as examples of large numbers, and the size of a cell in your body, a virus, or an atom as examples of very small things. A quick check in an encyclopedia gives the width of the observable part of the universe as about 92 billion light years where a light year is about 9.5 trillion kilometres. So to find the width of the universe in kilometres, you need to multiply 92 billion by 9.5 trillion. How would you do that?

A trillion is one thousand billion or a million million.

What is the biggest number you can put in your calculator directly?

Well, you might have reached for your calculator, but there's a problem: 92 billion is 92 000 000 000 and that is too big a number to enter into your calculator directly.

So what next? You may have worked the calculation out on paper or multiplied 92 by 9.5 to get 874 and deduced that the distance must therefore be 874 billion trillion kilometres or you may just have been bewildered by the enormity of the numbers involved! Clearly the skills and notation we have used so far are not particularly helpful here. However there is a way round this which builds on the work you did with powers in Chapter 1.

In Chapter 1, you discovered that some large numbers could be written using power notation. For example, 100 is the same as 10 × 10 and this can be written as 10^2. Here 10 is known as the **base number** and the 2 is the **power**. Similarly, 1000 is the same as 10 × 10 × 10 or 10^3, and so on.

Write each of the following numbers as a power of ten.

(a) 10 000 (b) 1 000 000 (c) 1 000 000 000

(d) What do you notice about the number of zeros in the original number and the power of ten?

Write each of the following numbers in decimal form and as a power of ten.

(e) There are about one hundred thousand hairs on an average human head.

(f) By 2050 the population of the earth may be about 10 billion people.

(g) In 1961, the French poet Raymond Queneau wrote a book called *A Hundred Thousand Billion Poems*.

Comment

(a) $10\,000 = 10 \times 10 \times 10 \times 10 = 10^4$.

(b) $1\,000\,000 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$.

(c) $1\,000\,000\,000 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^9$.

(d) The number of zeros in the original number is equal to the power of 10.

(e) One hundred thousand is $100\,000$ or 10^5.

(f) Ten billion is $10\,000\,000\,000$ or 10^{10}.

Can you explain why there were 10^{14} different sonnets?

(g) One hundred thousand billion is $100\,000\,000\,000\,000$ or 10^{14}. Queneau's book contained 10 sonnets, each with 14 lines. Each page, containing one sonnet, was cut into 14 strips with one line on each strip, so it was possible to combine lines from different sonnets to form a new sonnet. There are 10^{14} different ways of making a sonnet in this way. So, the title of the book was correct!

This idea can be extended to write other numbers in this form. For example, $6\,000\,000$ can be written as $6 \times 1\,000\,000$ or 6×10^6. In the same way, $6\,500\,000$ can be written as $6.5 \times 1\,000\,000$ or 6.5×10^6.

When a number is written in this form, that is where a number between 1 and 10 is then multiplied by an integral power of ten, it is said to be in **scientific notation** or **standard form**.

So, the scientific notation for $6\,500\,000$ is 6.5×10^6, not 65×10^5, because 65 does not lie between 1 and 10.

Scientific notation

A number written in scientific notation has the following form:

$$(1 \leq \text{number} < 10) \times 10^{\text{integer}}.$$

Notice that to write a number in scientific notation, you can start by writing down the number between 1 and 10. So for $130\,000$, this number is 1.3; then multiply this number by 10 repeatedly until you reach the number required.

Each time you multiply by 10, the power of ten increases by 1.

So, $130\,000 = 1.3 \times 10^5$ because you have to multiply 1.3 by 10 five times to get $130\,000$.

Activity 89 Numbers in scientific notation

Write the following numbers without using powers of 10.

(a) 2×10^4 (b) 3.82×10^8 (c) 9.3567×10^2

Write the following numbers in scientific notation. Which is the biggest?

(d) 92 billion (e) 400 trillion (f) 9 500 000 000 000

Comment

(a) $2 \times 10^4 = 2 \times 10 \times 10 \times 10 \times 10 = 20\,000$.

(b) $3.82 \times 10^8 = 3.82 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 382\,000\,000$.

(c) $9.3567 \times 10^2 = 9.3567 \times 10 \times 10 = 935.67$.

(d) 92 billion $= 92\,000\,000\,000 = 9.2 \times 10^{10}$.

(e) 400 trillion $= 400\,000\,000\,000\,000 = 4 \times 10^{14}$.

(f) $9\,500\,000\,000\,000 = 9.5 \times 10^{12}$.

Notice how using scientific notation enables you to compare the sizes of numbers quickly. The highest power of the three numbers in this activity is 14, so 400 trillion is the biggest number here.

How does this notation help in working out the width of the universe?

Consider what happens when you multiply two different powers of ten together, say 10^2 and 10^4.

$$10^2 \times 10^4 = (10 \times 10) \times (10 \times 10 \times 10 \times 10)$$
$$= 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1\,000\,000.$$

There are six tens multiplied together, so we can write this as 10^6.
So $10^2 \times 10^4 = 10^6$.

You can get the same result by adding the powers together:
$10^2 \times 10^4 = 10^{(2+4)} = 10^6$.

Try working out $2^3 \times 2^5$ the 'long way' and then by adding the powers together.

Do your answers agree?

The following rule applies in general.

> To multiply two numbers with the same base number, add the powers.

Activity 90 Multiplying numbers with the same base number

Work out the following, giving your answer first in power form and then in decimal form.

(a) $2^2 \times 2^4$ (b) $(^-3)^2 \times (^-3)^3$ (c) $10^4 \times 10^3$

Comment

(a) $2^2 \times 2^4 = 2^{(2+4)} = 2^6 = 64$.

(b) $(^-3)^2 \times (^-3)^3 = (^-3)^{(2+3)} = (^-3)^5 = {}^-243$.

(c) $10^4 \times 10^3 = 10^{(4+3)} = 10^7 = 10\,000\,000$.

You can now work out the width of the universe! Recall that the calculation needed is 92 billion × 9.5 trillion. First express both numbers in scientific notation: 92 billion = 9.2×10^{10} and 9.5 trillion = 9.5×10^{12}.

So the width is 92 billion × 9.5 trillion km = $9.2 \times 10^{10} \times 9.5 \times 10^{12}$ km. This calculation can be rearranged as $9.2 \times 9.5 \times 10^{10} \times 10^{12}$. Multiplying 9.2 by 9.5 gives 87.4. Then adding the powers gives $10^{10} \times 10^{12} = 10^{(10+12)} = 10^{22}$.

So the calculation is

$$
\begin{aligned}
9.2 \times 9.5 \times 10^{10} \times 10^{12} &= 87.4 \times 10^{22} \\
&= 8.74 \times 10^1 \times 10^{22} \\
&= 8.74 \times 10^{(1+22)} \\
&= 8.74 \times 10^{23}.
\end{aligned}
$$

Hence, the width of the universe is 8.74×10^{23} km.

Now work through Section 5.1 of Chapter 5 of the Calculator Booklet.

Before we can explore how to write numbers less than 1 in scientific notation we need to work out a rule for dividing two numbers with the same base number.

Consider $10^6 \div 10^2$. This can be written as a fraction, and then cancelled, as follows:

$$\frac{10^6}{10^2} = \frac{10\times10\times10\times10\times10\times10}{10\times10} = 10 \times 10 \times 10 \times 10 = 10^4.$$

So $10^6 \div 10^2 = 10^4$.

This time you can get the same result by **subtracting** the powers: $10^6 \div 10^2 = 10^{(6-2)} = 10^4$.

Try working out $3^5 \div 3^3$ the long way, and then by subtracting the powers. Are you convinced that the following rule works?

> To divide two numbers with the same base number, subtract the powers.

Zero as a power

Now what happens if you work out $10^3 \div 10^3$?

The 'long way' gives $10^3 \div 10^3 = \dfrac{10^3}{10^3} = \dfrac{10\times10\times10}{10\times10\times10} = \dfrac{1}{1} = 1$.

The powers rule gives $10^3 \div 10^3 = 10^{(3-3)} = 10^0$. But what does 10^0 mean? '10 multiplied by itself zero times' does not make sense; so 10^0 is defined as having the value 1. You can use a similar argument to show that any number to the power zero has a value of 1. For example, $2^0 = 1$, $3.25^0 = 1$ and $(^-6)^0 = 1$.

Negative powers

Now work out $10^2 \div 10^6$.

The 'long way' gives

$$10^2 \div 10^6 = \frac{10^2}{10^6} = \frac{10\times10}{10\times10\times10\times10\times10\times10} = \frac{1}{10\times10\times10\times10} = \frac{1}{10^4}$$

The rule gives $10^2 \div 10^6 = 10^{(2-6)} = 10^{-4}$. So 10^{-4} means $\frac{1}{10^4}$.

We read 10^{-4} as 'ten to the power minus four'. (Sometimes, the word 'power' is left out and we say 'ten to the minus four' instead.)

In the same way, 5^{-2} is said as 'five to the minus two' and means $\frac{1}{5^2}$ or $\frac{1}{5\times5}$ or $\frac{1}{25}$.

in Chapter 3 when dividing fractions

Alternatively, you can say that 5^{-2} is the **reciprocal** of 5^2.

Activity [illegible] Negative powers

(a) Without using your calculator, write the following numbers as fractions or whole numbers, as appropriate.

(i) 2^{-3}

(ii) 3^{-4}

(iii) 6^0

(iv) 10^{-6}

(b) Write the following as powers of 10.

(i) 0.01

(ii) 0.000 000 1

Comment

(a) (i) $2^{-3} = \frac{1}{2^3} = \frac{1}{2\times2\times2} = \frac{1}{8}$

(ii) $3^{-4} = \frac{1}{3^4} = \frac{1}{3\times3\times3\times3} = \frac{1}{81}$.

(iii) $6^0 = 1$.

(iv) $10^{-6} = \frac{1}{10^6} = \frac{1}{10\times10\times10\times10\times10\times10} = \frac{1}{1000000}$.

(b) (i) $0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$.

(ii) $0.0000001 = \frac{1}{10000000} = \frac{1}{10^7} = 10^{-7}$.

Without using your calculator, work out each of the following.

(a) $4^7 \div 4^5$ (b) $3^4 \div 3^7$ (c) $2^4 \div 2^{-2}$ (d) $10^{-2} \div 10^2$

Comment

(a) $4^7 \div 4^5 = 4^{(7-5)} = 4^2 = 4 \times 4 = 16$.

(b) $3^4 + 3^7 = 3^{(4-7)} = 3^{-3} = \frac{1}{3^3} = \frac{1}{3 \times 3 \times 3} = \frac{1}{27}$.

(c) $2^4 \div 2^{-2} = 2^{(4-(-2))} = 2^{4+2} = 2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$.

(d) $10^{-2} + 10^2 = 10^{(-2-2)} = 10^{-4} = \frac{1}{10^4} = \frac{1}{10 \times 10 \times 10 \times 10} = \frac{1}{10000} = 0.0001$.

The table below summarises the decimal form for powers of 10, together with the SI prefixes and their names for reference:

Number	0.000001	0.001	0.01	0.1	1	10	100	1000	1000000	1000000000
Power	10^{-6}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}	10^{6}	10^{9}
SI prefix	μ	m	c	d		da	h	k	M	G
Prefix name	micro	milli	centi-	deci		deca	hecto	kilo	mega-	giga

You can now write numbers less than 1 in scientific notation.

For example, $0.03 = \frac{3}{100} = \frac{3}{10 \times 10} = \frac{3}{10^2} = 3 \times 10^{-2}$

An alternative way to write a number less than 1 in scientific notation is first to write it down as a number between 1 and 10, in this case it is 3. Then divide this number by 10 repeatedly until you reach the number required. Each time you divide by 10, the power of ten reduces by 1. So here you have to divide 3 by 10 twice to get 0.03, so $0.03 = 3 \times 10^{-2}$.

To convert a number in scientific notation back into decimal form, write down the negative power of 10 as a fraction and then divide the numerator by the denominator.

For example: $5.72 \times 10^{-4} = \frac{5.72}{10^4} = \frac{5.72}{10000} = 5.72 \div 10000 = 0.000572$

Write down the following numbers as decimals.

(a) 3×10^{-4}

(b) 7.5×10^{-5}

(c) 1.46×10^{-3}

Write down the following numbers in scientific notation.

(d) 0.000 007

(e) 0.0742

(f) 0.000 000 009 8

Comment

(a) $3\times10^{-4} = \frac{3}{10^4} = \frac{3}{10\times10\times10\times10} = \frac{3}{10000} = 0.0003.$

(b) $7.5\times10^{-5} = \frac{7.5}{10^5} = \frac{7.5}{10\times10\times10\times10\times10} = 0.000075.$

(c) $1.46\times10^{-3} = \frac{1.46}{10^3} = \frac{1.46}{10\times10\times10} = 0.00146.$

(d) $0.000007 = \frac{7}{1000000} = \frac{7}{10^6} = 7\times10^{-6}.$

(e) $0.0742 = \frac{7.42}{100} = \frac{7.42}{10^2} = 7.42\times10^{-2}.$

(f) $0.0000000098 = \frac{9.8}{1000000000} = \frac{9.8}{10^9} = 9.8\times10^{-9}$

Now before we carry on, it is worth spending a few moments thinking about how scientific notation fits in with what you already know about decimals. In Section 2.1 of the Calculator booklet, you saw that when you multiplied a number by 10, the digits in the number moved one place to the left and when you divided a number by 10, the digits moved one place to the right. So, extending this idea, if you multiply a number by 10^3, the digits will move **3** places to the **left** to make the number bigger. For example, $4.5 \times 10^3 = 4500$. Similarly, if you multiply a number by 10^{-4} which is the same as dividing by 10^4, the digits will move **4** places to the **right** to make the number smaller. For example, $2 \times 10^{-4} = 0.0002$.

Now work through Section 5.2 of Chapter 5 of the Calculator booklet.

Roots

You can express square roots using power notation as well. Consider $4^{\frac{1}{2}} \times 4^{\frac{1}{2}}$. Adding the powers, gives $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^1 = 4$. This shows that when you multiply $4^{\frac{1}{2}}$ by itself, you get 4. In other words, $4^{\frac{1}{2}}$ is a square root of 4. Alternatively, $4^{\frac{1}{2}} = \pm\sqrt{4} = \pm2$.

Similarly, $27^{\frac{1}{3}}$ means the **cube** root of 27, i.e. the number which when multiplied by itself three times gives 27. Since $3 \times 3 \times 3 = 27$, we say that 3 is the cube root of 27 or $27^{\frac{1}{3}} = 3$.

Now work through Section 5.3 of Chapter 5 of the Calculator booklet.

This section has introduced some new notation and some new ways of manipulating numbers written in scientific notation. If you have not already done so, you may find it helpful to look back through this section now and summarise the key points that you have learned, before moving on to the next section. You may like to make some entries in your dictionary.

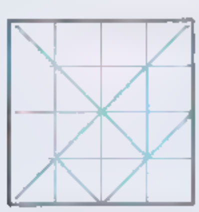
5.2 Scale diagrams

This section considers how you can use **scale diagrams** in some practical situations. Scale diagrams are particularly useful when you want to describe something accurately, but the life size version is either too big or too small to do so directly. For example, a map can highlight key features of the land and help you to plan a journey or a plan can indicate how to make something exactly. In a scale diagram, all the measurements on the diagram are a fraction (or a multiple) of the real measurements. For example, suppose you have a plan for a table and a scale of 1:50 is used. This means that the real measurements will be 50 times the measurements on the plan. So if a particular length on the plan is 4 cm, the corresponding length on the full-size table would be 4 cm × 50 = 200 cm or 2 m.

The earliest known book of tangram puzzles was published in China in 1813.

To understand how this works in practice, you can have a go at making a geometrical puzzle from a scale diagram in Activity 94. This kind of puzzle is known as a **tangram** and has been popular in many different countries for hundreds of years. It consists of seven geometrical pieces arranged in a square and the challenge is to try and arrange all seven pieces to make another shape or picture that is given to you. You will need the tangram for activities in Sections 3 and 4.

A scale diagram of the tangram puzzle is shown below. To make the puzzle easy to use, it needs to be enlarged in the ratio 1:5.



Measure the length of the side of the puzzle on the diagram. What is the corresponding length on the full-size puzzle?

The diagram has also been divided into a grid of 16 squares, each of side 0.5 cm. What size would these squares be on the full-size puzzle?

Now draw the square outline of the full-size puzzle on thin card. (You may find it helpful to stick a piece of graph paper onto the card first, so that your lines are straight and the corners square.) Add the grid of 16 squares to the puzzle by drawing the three horizontal and three vertical lines. Finally add the thick black construction lines.

Comment

All the lengths in the new puzzle are five times the corresponding length on the diagram. On the diagram, the length of the side of the puzzle is 2 cm. Since the ratio is 1:5, the length of the real puzzle is therefore 2 cm × 5 = 10 cm.

On the diagram, the length of each side of the squares in the grid is 0.5 cm.

The length of each side of a grid square on the puzzle will be 0.5 cm × 5 = 2.5 cm.

Keep your puzzle handy, but don't cut it up yet – there's more to be done!

You may have used scale diagrams yourself for DIY work in your house or garden. If you are planning a new look for a room or a garden, it helps to make a detailed plan so that you can see how things will fit together. Bathroom suppliers and DIY stores often provide planning grids and scaled cut-out shapes for the different items such as baths and cupboards, and you can then try out different arrangements on your plan without the upheaval of moving the furniture itself. A scale diagram also helps you to see what will fit and more importantly what won't and this can prevent expensive mistakes!

Imagine a planning grid marked in 0.5 cm squares with scale given as 1:20. What would each square on the plan represent in real life?

Since the scale is 1:20, the side of each of these squares will represent 0.5 cm × 20 or 10 cm.

In order to use such a grid, you will need to add the dimensions of the room and the measurements of fixtures like doors, windows and radiators yourself. For example, suppose you measure your bathroom and find that it is 3.5 metres long and 2.45 metres wide. So that the diagram will fit onto an A4 piece of paper, appropriate units to use for the diagram are either centimetres or millimetres. So first convert the real-life measurements into centimetres.

The length of the bathroom is 3.5 m = 3.5 × 100 cm = 350 cm.

The edge of each square on the plan represents 10 cm.

Since 350 ÷ 10 = 35, the edge of 35 squares will represent 350 cm.

The width of the bathroom is 2.45 m = 2.45 × 100 cm = 245 cm.

Since 245 ÷ 10 = 24.5, the edge of 24.5 squares will represent 245 cm.

On the planning grid provided separately, draw a rectangle 35 squares long and 24.5 squares wide to mark the boundary of the bathroom.

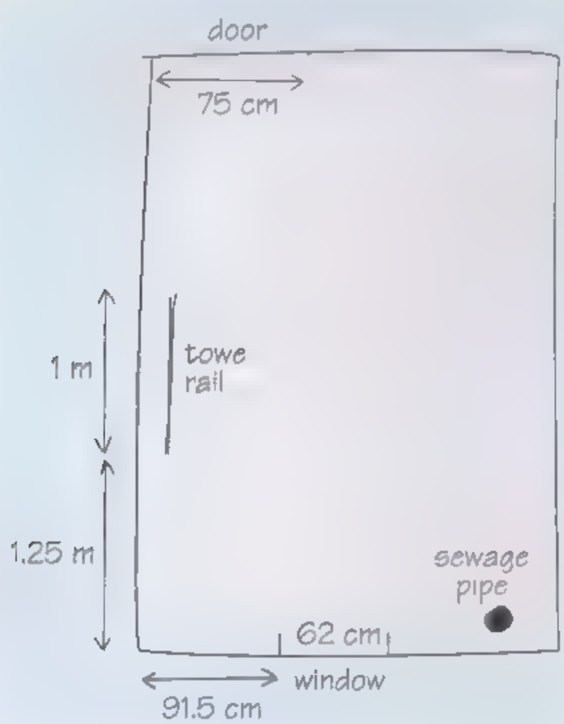
Alternatively, rather than counting squares, you can calculate the lengths on the plan directly. If the scale is 1:20, then you will have to divide both the length and the width of the bathroom by 20, to find the corresponding lengths on the plan. The length of the bathroom will be 350 cm ÷ 20 = 17.5 cm and the width of the bathroom will be 245 cm ÷ 20 = 12.25 cm.

Check that your rectangle does measure 17.5 cm by 12.25 cm!

Activity [illegible] Designing a bathroom

In the bathroom described above, the door is on the shorter wall in the corner, and it is 75 cm wide. A heated towel rail 1m wide is on the longer wall, 1.25 m from the corner of the bathroom. On the wall opposite the door, there is a window that is 62 cm wide and 91.5 cm from the corner (but at a height of 1.6 m from the floor).

Here is a rough sketch.



What will these measurements be on the plan?

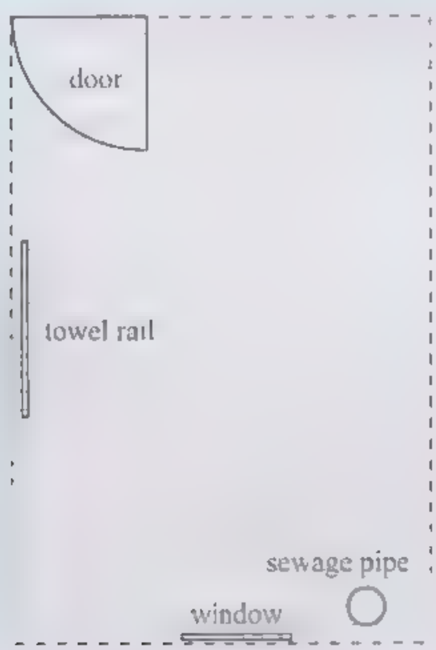
Mark the towel rail and window on the plan. How would you arrange the bath, toilet and basin in the bathroom?

Comment

The door is 75 cm wide and this will be represented by the edge of $75 \div 10 = 7.5$ squares or a length of $75 \text{ cm} \div 20 = 3.75$ cm on the plan.

The towel rail is 1m wide. This is the same as 100 cm, so it will be represented by the edge of $100 \div 10 = 10$ squares or a length of $100 \text{ cm} \div 20 = 5$ cm on the plan. It is at a distance of 1.25 m or 125 cm from the corner. This length will be represented by the edge of 12.5 squares or by $125 \text{ cm} \div 20 = 6.25$ cm.

The window is 62cm wide and this will be represented by the edge of 6.2 squares or by a length of $62 \text{ cm} \div 20 = 3.1$ cm on the plan. It is 91.5 cm from the corner. This length will be represented by the edge of 9.15 squares or by $91.5 \text{ cm} \div 20 = 4.575$ cm on the plan. You will have to round these measurements before drawing them on the plan. The diagram shows how the plan should look. (It is not to the scale given above.)



To arrange the bath, basin and toilet, cut out shapes drawn to scale and position them on the plan, allowing space to move around each item. For ease of installation, the toilet should be positioned next to the sewage pipe, but there are several possibilities for the bath and basin. If other items such as cupboards are required, you can make scaled shapes for these and add them to the plan.

These two examples, the tangram and the bathroom plan, illustrate how diagrams can be used to make practical problems easier to solve. If you are constructing your own scale diagram and have taken a lot of measurements,

you may find it clearer to tabulate your results as shown below. Why not try this for a room in your house?

Item	Length in real life/m	Length on plan/cm
Patio door	2.20	11

5.3 Geometry

When you are describing something geometrical, it helps to be familiar with some basic definitions and vocabulary relating to angles, lines and shapes. As you study this section, you may like to add any new definitions to your dictionary or your notes.

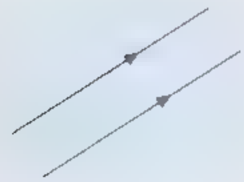
Angles and lines

Why 360°? We inherited this from the Babylonians! Their counting system was based on 60 and they first used degrees in astronomy.

Angles measure the amount of turning from one position to another. Imagine looking straight ahead and then turning round until you get back to your starting position. The angle you have turned through is a full turn. If you turn so that you are facing in the opposite direction, you will have made half of a full turn. If you turn from looking straight ahead to facing either directly to your right or directly to your left you will have made a quarter turn. Smaller turns can be described by splitting the full turn into 360 equal intervals known as degrees. So one full turn is the same as 360 degrees which is written as 360°. This means that half a turn is 180° and a quarter turn is 90° as shown below.

An angle of 90° is known as a **right angle**.

Lines which are at right angles to each other are said to be **perpendicular**. On a diagram right angles are denoted by a small square drawn at the angle. Angles can be measured using an instrument known as a **protractor**.



Parallel lines are always the same distance apart and, if you extended them indefinitely, they would never meet. A railway track is an example of a set of two parallel lines. If two lines are parallel, this can be shown on a diagram by drawing an arrow (or double arrow) on each line.

Shapes

Activity 9[illegible] Everyday shapes

You are probably already familiar with shapes such as triangles, squares, rectangles and circles from everyday life. Spend a few moments summarising what you know about these shapes already, including how you recognise them and where you might find them.

Comment

Triangles have three straight sides and are often used in buildings as they cannot be distorted. For example, they may be used for bracings or as supports.

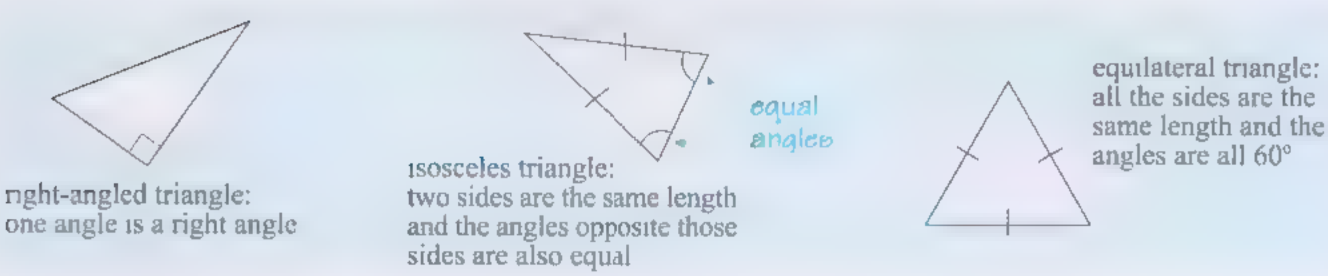
Squares and rectangles both have four straight sides and their angles are all right angles. In addition, the sides of a square are all the same length. These shapes can be seen everywhere – books, tables, tiles, buildings, … .

Why are inspection covers usually round? Hint: Think about the possibility of dropping the cover into the hole!

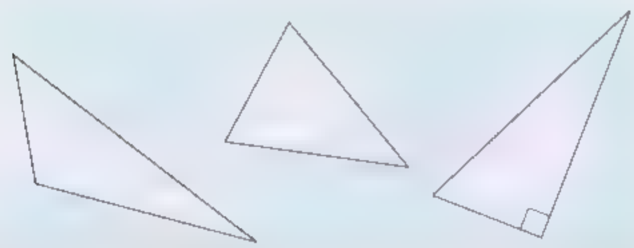
Circles can be drawn by specifying a centre point and a radius – the distance from the centre to the circle's edge. The radius is always a constant length. The diameter of a circle is the distance from one edge of the circle to the other, passing through the centre. The edge of the circle is known as the circumference. Circles are often used where movement is involved (for example, wheels) or where lack of corners is important (for example, cups or bowls).

You may have mentioned some other properties of these shapes as well.

Some special kinds of triangles are shown below. Sides that are the same length are marked with the same symbol, usually a short line, perpendicular to the side.



A triangle in which all the sides have different lengths is known as a **scalene** triangle; some examples are shown below.

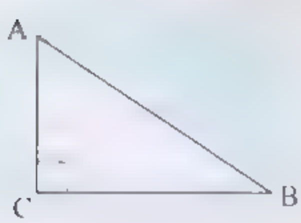


Another important fact is that the angles of a triangle add up to 180°. To illustrate this, if you cut out any triangle and then tear off the angles, you will be able to arrange them to form a straight line.

Shapes which have four straight edges are known as **quadrilaterals**. So squares and rectangles are special kinds of quadrilaterals.

A quadrilateral which has one set of parallel sides is known as a **trapezium**.

If the quadrilateral has two sets of parallel sides, it is called a **parallelogram**.

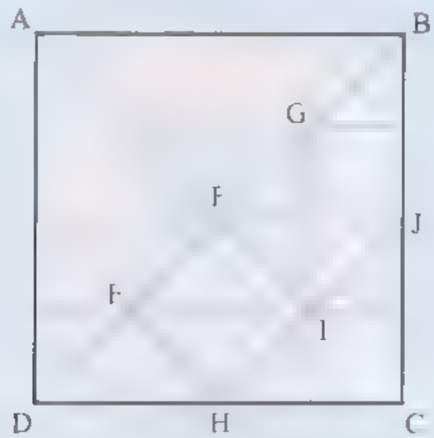


When you are describing a geometrical figure, you often need to refer to a particular line or angle on the diagram. This can be done by labelling the diagram with letters. For example, the diagram shows a triangle ABC, in which the longest side is AB and the angle $A\hat{C}B$ is a right angle. $A\hat{C}B$ is the angle formed by the lines AC and CB. The point where two lines meet is known as a **vertex** (plural, vertices). So A, B and C are **vertices** of the triangle.

Notice that you can use the shorthand notation 'ΔABC' for 'the triangle ABC', if you wish.

Activity 95: What can you see?

Find your tangram puzzle from Activity 94 and label it as shown.



Which sets of lines are parallel? Which lines are perpendicular? How many triangles can you see in the puzzle? What features are the same and what features are different in these triangles? What other shapes can you see?

Comment

There are four sets of parallel lines. AB is parallel to DC. This can be written as $AB \parallel DC$. Similarly, $DB \parallel HJ$ and $AI \parallel EH$. Also, AD, GI and BC are all parallel to each other.

AB is perpendicular to AD and to BC. This can also be written as $AB \perp AD$ and $AB \perp BC$. Similarly $DC \perp AD$ and $DC \perp BC$. $EH \perp DB$, $EH \perp HJ$, $AI \perp DB$ and $AI \perp HJ$.

There are the seven triangles in the puzzle: ΔABD, ΔDBC, ΔABF, ΔADF, ΔDEH, ΔFGI and ΔHJC – give yourself a pat on the back if you spotted them all!

If you did not spot all of these, trace out the triangles by following the letters.

All these triangles are right angled. In each triangle there is one right angle of 90° and then two other angles which are both half of 90° which is 45°. If two figures have exactly the same shape, but not necessarily the same size, the two figures are said to be **similar**. One figure is an enlargement of the other. For example, ΔAFD is similar to ΔFGI because the sides of ΔAFD are all double the corresponding sides of ΔFGI. Here, all the triangles in the puzzle are similar to each other. In general, two figures are similar if corresponding sides are in the same ratio and corresponding angles are equal. So trapezium $FBJI$ is similar to the trapezium $JCDF$.

Note how the mathematical meaning of the word 'similar' is different from and more precise than the everyday meaning.

In each triangle, two of the angles are the same, so the triangles are isosceles. You may also have spotted this by noting that each triangle has two sides the same length.

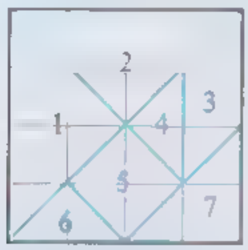
Can you spot any other congruent triangles in the puzzle?

Some of the triangles such as ΔDEH and ΔFGI are exactly the same size and shape and are said to be **congruent** – one would lie exactly on top of the other one.

Trapezia is the plural of trapezium.

There are many other shapes in the diagram. Did you spot the parallelogram, *GBJI*, the squares *ABCD* and *EFIH* and the trapezia, *DHIF*, *EHIG*, *DBJH*, *BJHE* and *DGIH*?

Now, before you cut up the puzzle, label the pieces as shown below.



Now, cut along the thick lines on the puzzle to make seven pieces. It's worth finding an envelope or plastic wallet to keep the pieces in when you are not using them.

Activity 9 Making shapes

Cover up the comments below before you start!

Find the square and the two smallest triangles (shapes 4, 5 and 6). Using these three shapes only, make:

(a) a rectangle (b) a triangle (c) a parallelogram.

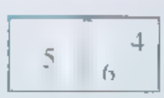
Can you make a square using five, six and seven pieces too?

Make a square using:

(d) 2 pieces (e) 3 pieces (f) 4 pieces.

Comment

(a)



(b)

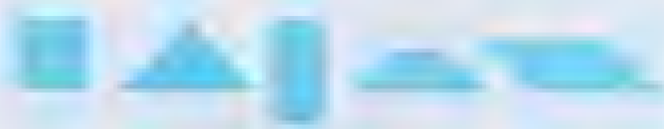
(c)

(d)

(e)

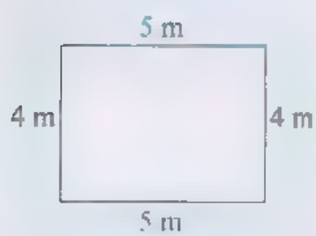
(f)

Using all seven tangram pieces for each shape, can you make the following shapes? What other shapes or pictures can you make?



This section has introduced a lot of mathematical vocabulary, which you will need in the later sections of this chapter. As well as making notes on these terms and/or adding them to your dictionary, it is worth looking back over these definitions frequently – say at the start of each study session so that they become familiar to you. Try to practise using the terms elsewhere in your everyday life as well – can you spot any isosceles triangles, parallel lines or similar shapes anywhere?

5.4 Perimeters



Imagine you were decorating a room and you decided to put a border all round the top of the walls. To work out the length of the border, you would need to measure the lengths of the walls and add these measurements together. For example, if the room is rectangular and measures 5 m by 4 m, the length of the border would be 5 m + 4 m + 5 m + 4 m or 18 m, as shown in the diagram.

Make sure all the lengths are measured in the same unit

The distance around the edge of a shape is known as the perimeter. The prefix 'peri' means around and 'meter' means measure, so 'perimeter' means measuring around a shape. For a shape with straight edges, you can work out the perimeter by measuring the length of each edge and then adding these lengths together.

Using all seven tangram pieces, make one shape which has the largest perimeter and one shape which has the smallest perimeter. Can you prove your results?

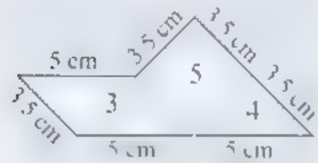
By measuring lengths on your tangram puzzle, work out the perimeter of the following:

(a) the square *ABCD*

(b) ΔFGI

(c) the trapezium *DGIH*

(d) this figure formed from shapes 3, 4 and 5:



Comment

(a) *ABCD* is a square whose sides are 10 cm. So the perimeter is
$10\text{ cm} + 10\text{ cm} + 10\text{ cm} + 10\text{ cm} = 40\text{ cm}$.

(b) In the ΔFGI, $GI = 5$ cm, $GF \approx 3.5$ cm and $FI \approx 3.5$ cm. Adding these lengths together gives the perimeter as approximately 12 cm.

(c) *DH* and *GI* are each 5 cm long; *DG* is approximately 10.6 cm and *HI* is approximately 3.5 cm. So the perimeter of *DGIH* is
$10.6\text{ cm} + 5\text{ cm} + 3.5\text{ cm} + 5\text{ cm} = 24.1\text{ cm}$.

(d) Starting on the left at the top of shape 3 and going clockwise round the figure, the perimeter is approximately:

$5\text{ cm} + 3.5\text{ cm} + 3.5\text{ cm} + 3.5\text{ cm} + 5\text{ cm} + 5\text{ cm} + 3.5\text{ cm} = 29\text{ cm}$.

When a shape has straight sides, you can measure the perimeter fairly easily, by considering each side in turn. However, measuring lines that are not straight can be more difficult. If you need the measurement for some practical purpose, then you can use a piece of string to wrap round an object and then measure the string.

Activity 10[illegible] Circles

You will need a tape measure (or a piece of string), a 30-cm ruler and five objects such as mugs, tins, bowls or buckets which are cylindrical with circular tops. Measure the circumference and diameter of each object in centimetres to the nearest 0.1 cm. Use the tape measure to find the circumference and the ruler to find the diameter. If you do not have a tape measure, use a piece of string and then measure the string using the ruler. You could also use a strip of graph paper to measure the circumference. Write down the diameter and the circumference of each object in a table and calculate the ratio $\dfrac{\text{circumference}}{\text{diameter}}$. What do you notice?

Comment

The results I collected are shown below.

Item	Diameter in cm	Circumference in cm	$\frac{\text{circumference}}{\text{diameter}}$ (to 1 d.p.)
Spice jar	4.4	14.1	3.2
Glass	6.6	20.9	3.2
Tin	8.6	27	3.1
Mug	10.3	32.3	3.1
Bowl	23.0	72.8	3.2

It is quite difficult to measure these quantities accurately. However it is noticeable that in each case, the ratio of the circumference to the diameter seems to be about 3.1 or 3.2. In other words, the circumference is just over three times the length of the diameter.

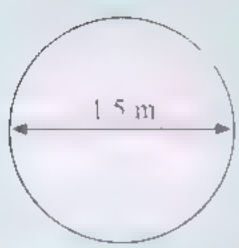
Pi occurs in [illegible] branches of mathematics, [illegible] just problems with circles.

If you could measure these objects more accurately, you would find that the circumference divided by the diameter always gives exactly the same answer. This value is known as 'pi' (pronounced 'pie') and it is denoted by the Greek letter π. The value of π is approximately 3.142, although for most calculations, you will be using the π button on your calculator.

The following formulas enable you to work out the circumference of any circle.

> Circumference = π × diameter or circumference = 2 × π × radius.

For example, suppose a circular table has a diameter of 1.5 metres. How many people can sit down comfortably for a meal at the table?



The circumference of the table is given by the formula:

circumference = π × diameter.

As we just require a rough estimate here, we can substitute 3.142 for π and 1.5 for the diameter. This gives:

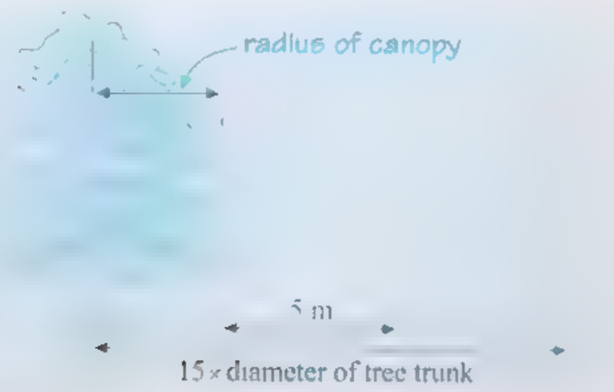
circumference of the table ≈ π × 1.5 m = 4.7 m (to 1 d.p.).

Hence the circumference is about 4.7 m.

Assuming each person needs a space of about 0.75 m in width, the number of people who can fit around the table is 4.7 ÷ 0.75. So six people should be able to fit around the table.

Now work through Section 5.4 of Chapter 5 in the Calculator booklet.

The next example is concerned with preserving old trees. In my local park, there is a large old beech tree. To help preserve old trees, the Woodland Trust (2004) recommends establishing a circular root protection zone with the tree at the centre. The radius of this circle should be either 15 times the diameter of the tree trunk or 5 m beyond the canopy, whichever is greater.



To find the diameter of the tree trunk, it is easiest to measure the circumference of the tree trunk at chest height and then work out the diameter of the tree from that. The circumference can also be used to get an estimate of the age of the tree.

Activity 101 Protecting an old tree

The circumference of the tree at chest height is 3.05 metres.

(a) The canopy has a radius of 12 m. What is the radius of the circle for the root protection zone, based on this measurement? If a wire fence is to be put up around this circular boundary, what length of fencing is needed?

(b) In the UK, beech trees are estimated to increase their trunk circumference by between 1.5 cm and 2 cm each year. Assuming this is true, how old is the beech tree?

(c) By drawing a 'doing–undoing' diagram, work out a formula for the diameter of a circle if you know the circumference. What is the diameter of the tree trunk?

(d) Based on your answers to parts (a) and (c), what radius should be used for the circular root protection zone?

Comment

(a) The radius will be 12 m + 5 m = 17 m. The circumference of a circle of radius 17 m is $2 \times \pi \times 17$ m = 106.8 m (to 1 d.p.).

So, a 107 m length of fencing should be sufficient.

(b) Since the growth is given in cm, change the circumference measurement into cm first: 3.05 m = 3.05×100 cm = 305 cm. If the circumference increases by 1.5 cm a year, the tree will be $305 \div 1.5 \approx 203$ years old. If the circumference increases by 2 cm a year, the tree will be $305 \div 2 = 153$ years old. The growth will depend on the climate and other factors, so a reasonable estimate for the age of the tree is 150 to 200 years old.

If the circumference increases by 2 cm every year, can you work out the increase in the diameter?

(c) To find the circumference, you multiply the diameter by π.

Doing

diameter ⟶ $\boxed{\text{multiply by } \pi}$ ⟶ circumference

So to find the diameter, you need to undo this step.

Undoing

diameter ⟵ $\boxed{\text{divide by } \pi}$ ⟵ circumference

Hence, the diameter is $305 \text{ cm} \div \pi \approx 97 \text{ cm}$ (to 2 s.f.).

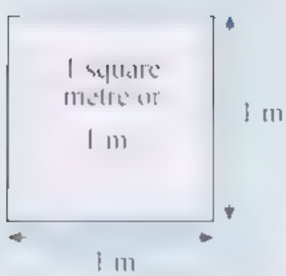
(d) 5 m beyond the canopy gives a radius of $12 \text{ m} + 5 \text{ m} = 17 \text{ m}$.

Fifteen times the diameter of the trunk $\approx 15 \times 97 \text{ cm} = 1500 \text{ cm}$ (to 2 s.f.).

Converting back into m, $1500 \text{ cm} = 1500 \div 100 \text{ m} = 15 \text{ m}$.

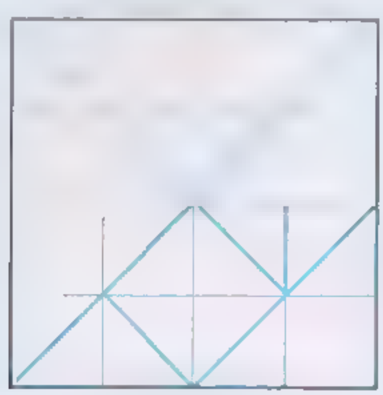
Since $17 > 15$, the root protection zone should have a radius of 17 m.

5.5 Areas



If you are planning to paint a wall, one of the first questions to ask is how much paint will you need and this will obviously depend on the size of the wall. Tins of paint usually state the area that the paint will cover. For example, on the back of a 2.5 litre tin of emulsion paint, it says that the paint will cover 'up to 35 square metres'. A square metre is the area that is covered by a square whose sides are 1 m long. This can also written as 1 m^2. So 35 m^2 will be the same area as the area of 35 of these 1 metre squares.

Areas can also be measured in other square units, such as square centimetres (cm^2) or square kilometres (km^2), depending on what is appropriate to the situation. A square centimetre is a square whose sides are 1 cm long and a square kilometre has sides which are 1 km long. The tangram puzzle below is drawn on a grid in which the lines are 1 cm apart so each grid square has an area of 1 cm^2. Since the whole puzzle measures 4 cm by 4 cm, its area is 16 cm^2 (you can confirm this by counting the grid squares on the puzzle).



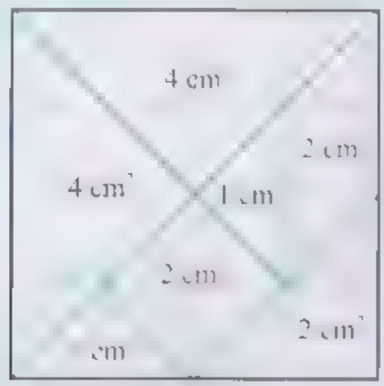
The 4 cm tangram

Activity 102 Counting squares

All the shapes of the puzzle are made up out of whole squares or half squares, so to find the area of any of the shapes, you can count the squares. **Using the 4 cm tangram** above, work out the area of each shape in square centimetres. Write the area on each shape.

Comment

The areas in square centimetres are shown below.



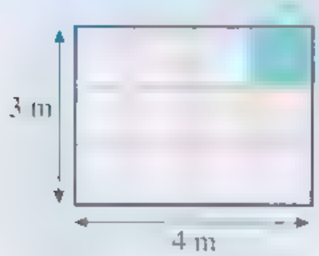
It's always a good idea to check your answer a different way if you can!

As a check you can add the individual areas together and see if they add up to the total area of 16 cm².

The sum of the individual areas in cm² is:

$4 + 4 + 1 + 2 + 1 + 2 + 2 = 16$. ✓

If you do need to calculate the area of some shape, laying a square grid (marked with m, cm or mm squares) over the shape and then counting the number of squares is one way to do it. This is particularly useful if the shape is irregular, for example a handprint or footprint.



However many areas can be built up out of basic shapes such as rectangles, triangles or circles and these areas can be calculated by using formulas. For example, if a rectangular room measures 4 m by 3 m and you are covering it in carpet tiles which are each 1 metre square, the tiles can be arranged in 3 rows each with 4 tiles in. So the total number of tiles will be $3 \times 4 = 12$ and the area covered is 12 m².

This area could have been calculated directly by multiplying the length of the room by its width.

Provided both measurements are in the same units, the following formula holds for any rectangle.

> Area of a rectangle = length × width.

For example, if the room measured 3.24 m by 4.38 m, the floor area would be

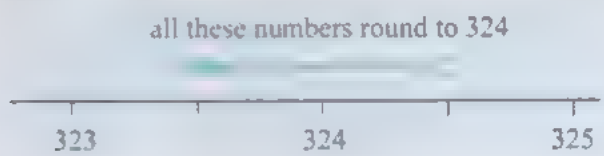
$3.24 \text{ m} \times 4.38 \text{ m} = 14.1912 \text{ m}^2 = 14.2 \text{ m}^2$ (to 3 s.f.).

Did you notice that the answer has been rounded?

Any measurement that you make is rounded – for example, lengths may be measured to the nearest mm or nearest cm depending both on the context and the instrument you are using to make the measurement. In the example above, the measurements have been made to the nearest centimetre. This means that the answer cannot be given to more than the nearest square centimetre under any circumstances and it is likely that the precision of the answer will be considerably less than that as the analysis below shows.

The width of the room has been measured as 324 centimetres.

But all lengths from 323.5 cm up to, but not including, 324.5 cm would be rounded to 324 cm.



Similarly, the length has been measured as 438 cm and all measurements from 437.5 cm up to, but not including, 438.5 cm would be rounded to 438 cm.

This means that the smallest possible area of the room is
$4.375 \text{ m} \times 3.235 \text{ m} = 14.153\,125 \text{ m}^2$.

The largest possible area of the room is $4.385 \text{ m} \times 3.245 \text{ m} = 14.229\,325 \text{ m}^2$.

These two areas only agree if they are rounded to three significant figures.

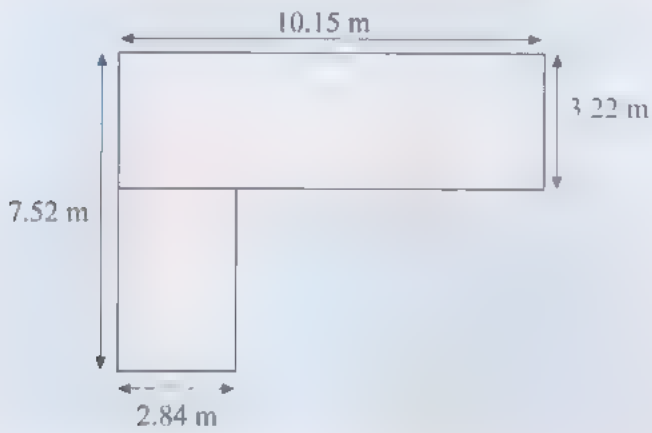
So, the area of the room is 14.2 m^2.

> If a calculation involves measurements, remember to round your answer appropriately at the end of your calculation.

The answer should be no more accurate than the least accurate number you have used. Some people use the rule of thumb 'round the answer to one significant figure less than the least accurate number in the calculation'. However, this is not always appropriate. What would happen if you had one number in the calculation which was measured to one significant figure?

(a) A rectangular garden is 10.5 m long and 14.2 m wide. What is its area?

(b) What is the floor area of this L-shaped room?



Comment

(a) The area of the garden = 10.5 m × 14.2 m = 149.1 m^2.

So to 2 s.f. the area is about 150 square metres.

(b) The area of the room can be calculated by adding the areas of the two rectangles together.

The area of the top rectangle is 10.15 m × 3.22 m = 32.683 m^2.

The bottom rectangle has a width of 2.84 m.

Its length is 7.52 m − 3.22 m = 4.3 m.

So its area is 4.3 m × 2.84 m = 12.212 m^2.

Hence the total area is 32.683 m^2 + 12.212 m^2 = 44.895 m^2.

To 2 s.f. the total area of the room is 45 m^2.

Imagine a rectangle that is 18 cm long and 2 cm wide.

(a) What is the area of the rectangle? What is its perimeter?

(b) Can you draw another rectangle which has the same area but a smaller perimeter? What is the smallest perimeter you can make?

(c) Can you draw another rectangle which has the same perimeter as in part (a) but a smaller area?

Comment

(a) The area of the rectangle is 18 cm × 2 cm = 36 cm^2.

The perimeter is 18 cm + 2 cm + 18 cm + 2 cm = 40 cm.

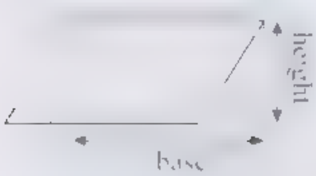
(b) The area must be 36 cm², and the perimeter must be less than 40 cm. So we want to find two numbers which multiply together to give 36 and which add together to give a number less than 20. There are many possibilities, for example a rectangle that is 9 cm by 4 cm has an area of 36 cm², but its perimeter is only 26 cm. Similarly, a rectangle with sides 10 cm and 3.6 cm has the same area, but a perimeter of 27.2 cm. The smallest perimeter (24 cm) is for a square of side 6 cm.

(c) Here we need to find two numbers which add up to 20 cm, but whose product is less than 36. A rectangle with sides of 19 cm and 1 cm has a perimeter of 40 cm but an area of only 19 cm². Another rectangle would be one with sides of 19.5 cm and 0.5 cm which has an area of 9.75 cm². Other combinations are possible.

If you know the formula for the area of a rectangle, you can work out formulas for the area of a parallelogram and a triangle.

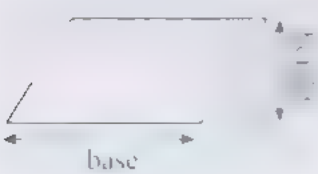
Take a parallelogram and cut off the left-hand edge and stick it next to the right-hand edge to make a rectangle. This has the same area as the parallelogram.

As the area of the rectangle can be found by multiplying its base by its height.



Area of a parallelogram = base × height.

Now if you cut this parallelogram in half along a diagonal, there are two possibilities:

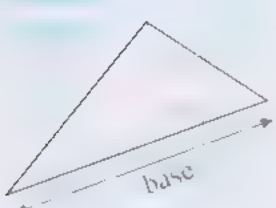


In each case the parallelogram has been split into two triangles which have the same area. So the area of each triangle is half the area of the parallelogram.

This gives a general formula for the area of a triangle:

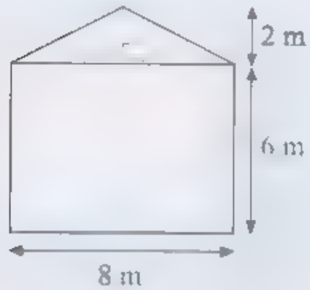
Area of a triangle = $\frac{1}{2}$ × base × height.

The height goes through the vertex that is opposite the base and is always perpendicular to the base.



To find the area of a triangle quickly, you can work out half the base and multiply it by the height (or vice versa). Or you can multiply the base by the height and then divide by 2.

The diagram below is a rough sketch of the gable end of a house that needs weatherproofing.



To work out the quantity of materials required, the area of the wall is needed. We can break this problem down by splitting the area into a rectangle and a triangle, then working out these areas and finally adding the two areas together to get the total. Assume the measurements have been made to the nearest 10 cm, that is to 2 s.f.

The area of the rectangle = length × width = $8 \text{ m} \times 6 \text{ m} = 48 \text{ m}^2$.

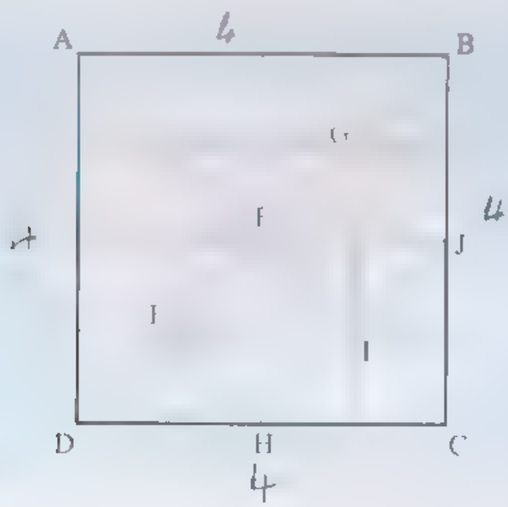
The triangle has a base of length 8 m and a perpendicular height of 2 m.

So the area of the triangle is $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 8 \text{ m} \times 2 \text{ m} = 8 \text{ m}^2$.

Hence the total area of the gable end is $48 \text{ m}^2 + 8 \text{ m}^2 = 56 \text{ m}^2$ or approximately 60 m^2 (rounding to 1 s.f.).

[illegible]

The figure below shows a 4 cm square tangram puzzle.



By using the formulas for the area of a triangle and for the area of a parallelogram, calculate the areas of the following shapes.

(a) ΔAFD with AD as the base.

(b) ΔDEH with DH as the base.

(c) Parallelogram $GBJI$ with BJ as the base.

(d) ΔJHC.

Check that your answers agree with those you obtained by counting the squares directly in Activity 102.

Comment

(a) $AD = 4$ cm. The perpendicular height from F onto $AD = 2$ cm.

The area of a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$.

So the area of $\Delta AFD = \frac{1}{2} \times 4 \text{ cm} \times 2 \text{ cm} = 4 \text{ cm}^2$.

How many other ways could you calculate this area?

(b) $DH = 2$ cm. The perpendicular height from E onto $DH = 1$ cm.

So the area of $\Delta DEH = \frac{1}{2} \times 2 \text{ cm} \times 1 \text{ cm} = 1 \text{ cm}^2$.

(c) $BJ = 2$ cm. The perpendicular height from G onto $BJ = 1$ cm.

The area of a parallelogram = base × height.

Hence, the area of $GBJI = 2 \text{ cm} \times 1 \text{ cm} = 2 \text{ cm}^2$.

(d) The base $HC = 2$ cm and the corresponding height $JC = 2$ cm.

So, the area of $\Delta JHC = \frac{1}{2} \times 2 \text{ cm} \times 2 \text{ cm} = 2 \text{ cm}^2$.

Many area problems can be sorted out by using combinations of squares, rectangles and triangles. However, you often need to find circular areas too. The formula for the area of a circle is:

> Area of a circle = $\pi \times (\text{radius})^2$.

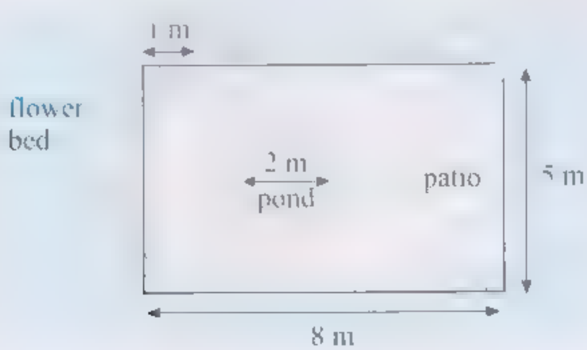
Now work through Section 5.5 of Chapter 5 in the Calculator booklet.

How would you find the surface area of a box or a cylinder?

Now that you know how to calculate the areas of basic shapes, you can calculate much more complicated areas by breaking each shape into basic shapes and adding the individual areas together. Even the surface areas of some containers which have curved surfaces like cylinders can be broken down into circles and rectangles.

Activity 1[illegible] Making a garden

The sketch shows a rough plan of a garden which contains a circular pond, a semicircular patio and two flower beds with circular front edges.



The remaining part of the garden is to be covered in lawn turf. If turf is sold in 1 m^2 rolls, how many rolls will be needed? If you get stuck, try using some of the strategies we used for the insulation problem in Chapter 3 such as I know I want and taking small steps.

Comment

We want to work out how many rolls of turf are needed and that means estimating the area of the lawn in square metres. Working to two decimal places throughout the calculation will provide sufficient precision here. The patio, pond and flower beds will not be turfed, so we need to subtract these areas from the total area of the garden.

The garden is a rectangle, so the area = length × width = 8 m × 5 m = 40 m^2.

The diameter of the pond is 2 m, so its radius is 1 m.

The area of a circle = $\pi \times (\text{radius})^2 = \pi \times 1^2\ m^2 \approx 3.14\ m^2$.

The radius of the patio is 5 m ÷ 2 = 2.5 m. The area of a circle of radius 2.5 m is $\pi \times 2.5^2\ m^2 \approx 19.63\ m^2$. The semicircular patio is half this area, which is $19.63\ m^2 \div 2 \approx 9.82\ m^2$.

The flower beds are each quarter of a circle of radius 1 m. The area of a circle of radius 1 m is $\pi \times 1^2\ m^2 = 3.14\ m^2$. So two quarter circles have a total area of $2 \times 3.14 \div 4\ m^2 = 1.57\ m^2$.

The lawn area = area of garden − area of pond − area of patio − areas of beds.

Hence, lawn area in $m^2 \approx 40 - 3.14 - 9.82 - 1.57 = 25.47$.

Hence, 26 m^2 of turf or 26 rolls will be needed. Note that if you wished to calculate the area of the lawn more precisely, you would need to use full calculator accuracy in the calculations. However, this estimate is sufficient for this situation.

Converting between area units

In the previous section we stressed the importance of using the same units all the way through a calculation. However there may be occasions when you need to convert from one square unit to another. For example if you want to find the area of a piece of land from a map, you may measure the area in square centimetres on the map, but this would be an inappropriate unit for the amount of land itself. To change from one square unit to another, you need to know how many of the smaller units are equivalent to one of the larger units. For example, how many square millimetres are needed to cover one square centimetre?

If you are not sure, a diagram might help.

If you draw a square centimetre and imagine covering it with square tiles which each have an area of 1 millimetre square (1 mm^2), you would be able to fit 10 of the 1 mm square tiles along the bottom edge of the 1 cm square tile, since there are 10 mm in 1 cm. Then you would be able to fit nine further rows each of 10 1 mm square tiles next to the first row. So 10 rows each with 10 1 mm square tiles would be needed to cover 1 square centimetre.

In other words, there are 100 square millimetres in 1 square centimetre:

$1\ cm^2 = 10\ mm \times 10\ mm = 10 \times 10\ mm^2 = 100\ mm^2$.

[illegible]

(a) Explain how many square centimetres are equivalent to one square metre.

(b) Convert 6000 cm^2 into m^2.

(c) Convert 0.85 cm^2 into mm^2.

(d) How many square metres are equivalent to 1 square kilometre?

Comment

(a) One square metre is 100 cm long and 100 cm wide. So we could fit 100 one centimetre square tiles along the edge of a square metre tile. But you could then fit 99 more rows of the small tiles above this row. So altogether there would be 100 rows of 100 small tiles or 10000 small tiles. In other words, $1\ m^2 = 10000\ cm^2$.

(b) Since there are 10000 cm^2 in 1 m^2 and we are converting from a small unit to a larger one, $6000\ cm^2 = 6000 \div 10000\ m^2 = 0.6\ m^2$.

(c) Since there are 100 mm^2 in 1 cm^2 and we are converting from a large unit to a smaller one $0.85\ cm^2 = 0.85 \times 100\ mm^2 = 85\ mm^2$.

(d) Since there are 1000 m in 1 km, 1 square kilometre is the same as 1000×1000 metres or $1\ km^2 = 1000000\ m^2$.

A hectare (abbreviation: ha) is defined as the area of a square 100 m by 100 m. So 1 ha = 10000 m^2. Alternatively, since 100 m = 0.1 km, a hectare is the same as 0.01 km^2.

The relationships between these different areas (km^2, ha, m^2, cm^2, mm^2) can be summarised by using a diagram similar to those we used in Chapter 2.

For example, suppose an area of lawn marked on a garden plan is approximately rectangular and measures 2 cm by 3.5 cm. If the scale of the map is 1:200, what is the actual area of the lawn?

If the scale is 1:200, the lengths on the ground will be 200 times the lengths on the plan.

Why is it not usually possible to estimate large areas (e.g. the area of a country) from, say, a world map?

So, the width of the lawn will be 2×200 cm = 400 cm = 4 m.

The length of the lawn will be 3.5×200 cm = 700 cm = 7 m.

Hence the area of the lawn = 4 m × 7 m = 28 m^2.

Notice that you would have obtained exactly the same answer by keeping the lengths in centimetres, calculating the area in cm^2 and then converting to m^2.

Area of the lawn = 400 cm × 700 cm = 280000 cm^2.

To convert this area to m^2, divide by the conversion factor of 10000, so area of lawn = 280000 ÷ 10000 m^2 = 28 m^2, as before.

An acre was originally defined as the area that could be ploughed in a day by a yoke of oxen. It is 4840 square yards.

The imperial measure of large area is the acre. If you look at advertisements for houses in a local paper you will probably find that some properties have information about the size of the plot or garden. Normally these are in terms of acres with gardens being described as $\frac{1}{4}$ acre, 1 acre or 3 acres. But how large are these areas? It sometimes helps to compare a new measurement with something you can visualise. For example, a standard adult football pitch is approximately 1.5 acres. So if you want to set up a football pitch in your garden, you now know how much space you need! An acre is about 0.4 hectare, and a hectare is just under 2.5 acres.

The ability to calculate or estimate areas occurs in many different practical situations from estimating the area of a piece of land or the area of a floor or wall in a room, to minimising the amount of packaging needed in a business and many other scientific and engineering applications.

In medicine it is important to be able to estimate the surface area of different parts of the body both for the treatment of burns and also for calculating some drug doses for chemotherapy. For example, if a burn covers more than 15 per cent of the body, the patient may suffer from shock and it is then important to get medical attention urgently.

But how do you do estimate a surface area as complicated as a human body?

[illegible]

How could you estimate the area of part of a body or a whole body? Think of as many approaches as you can, mathematical or otherwise and be creative! You don't need to work these estimates out at this stage – just make some suggestions on how you might do it.

In an emergency, how could you tell quickly whether a person had suffered burns on more than 15 per cent of their body?

Comment

One approach (that uses the mathematical modelling cycle from Chapter 2) would be to approximate the body by some simpler shapes such as cylinders and spheres and then, after taking quite a lot of individual measurements, work out the area of these shapes (for example, the surface area of a leg could be modelled by two cylinders). The total surface area of the body could then be estimated by adding the areas of the individual shapes together. However, there are many other ways of approaching this problem and some of these are discussed below. Which one you choose will depend on how accurate you need the answer to be and other physical considerations.

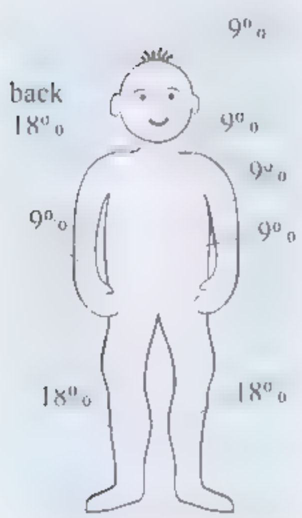
If there were some medical emergency, coming up with an answer one week later such as: 'The area burnt is 0.985 62 square metres and, as this is over 15 per cent of the body area, I recommend immediate medical attention' isn't going to help. It might be more accurate than a quick rough estimate but it is not what would be needed. A rough estimate will be good enough to make the decision about treatment.

A more practical approach of solving the problem might be to wrap the body in cloth so that it covered the skin, then remove the cloth and measure the areas of the flat cloth shapes that result.

Or you may decide to do some research, and see if there are any formulas for body area that might help.

Alternatively, the area of one side of a person's hand is reckoned to be about 1 per cent of their total body surface area. So you could draw round the hand, on graph paper marked in square centimetres, and then count the squares to find the area or just estimate this area by using simple shapes. This would give an estimate for the area of the hand and multiplying by 100 would then give an estimate of the total body area.

You could also use the hand area as an informal measure to find the area of a burn on part of the body by estimating how many handprints would cover the burn. For example, if the burn were covered by three handprints, an estimate for the area would be 3 per cent of the total body surface area. This rough estimate could be made very quickly at the scene of the accident. Although it may not be as accurate as the earlier methods, it would enable the paramedics to make a quick decision on the urgency of the situation and possible treatments.

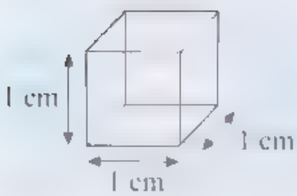


Paramedics also use the 'Rule of nines' for extensive burns on adults. This splits the body into sections with each section being either 9 per cent (head, chest, abdomen, arm) or 18 per cent, (leg, back) of the total body surface area. Although more accurate charts for estimating different body areas do exist, the Rule of nines is easy to remember and allows an estimate to be made quickly in an emergency.

5.6 Volumes

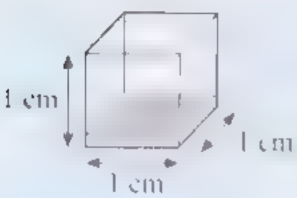
So far we have considered measuring lengths and areas, but most things in life are not flat!

Questions like 'How much does that hold?' need you to be able to specify the **volume** of an object. To do this, we can extend the ideas you met earlier. To specify how much space something needs or occupies, you can count how many **cubes** of a certain size will fit into the space, in the same way that you counted how many square tiles covered an area. A cube is the same shape as a box, but all its sides are the same length and its six faces are all square.



Useful cubes to use are the cubic millimetre (written as mm^3), the cubic centimetre (written as cm^3) and the cubic metre (written as m^3).

In each case, the length of the side of the cube is 1 unit. So a cubic centimetre has all its sides of length 1 cm, as shown.



Now imagine filling a cubic centimetre with cubic millimetres. Since there are 10 mm in 1 cm, a cubic centimetre will contain 10 layers with each layer made up of 10 rows, each of 10 cubic millimetres.
So $1\ cm^3 = 10 \times 10 \times 10\ mm^3 = 1000\ mm^3$. In other words, to convert a measurement in cubic centimetres into cubic millimetres, you would multiply by 1000. This makes sense – a cubic centimetre is larger than a cubic millimetre, so you would expect to need a lot more cubic millimetres to fill the same space.

How many cubic centimetres are there in a cubic metre?

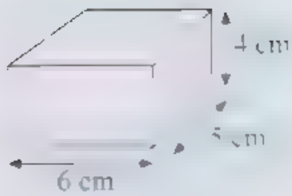
Comment

There are 100 centimetres in a metre. Imagine a cubic metre – it will be 100 cm long, 100 cm wide and 100 cm high. So you would be able to fit $100 \times 100 \times 100$ cubic centimetres into the cubic metre.

In other words, $1\ m^3 = 100 \times 100 \times 100\ cm^3 = 1\,000\,000\ cm^3$.

In Chapter 2, you met a different way of measuring volumes of liquids, namely millilitres, centilitres and litres. These units are linked to the cubic units because 1 ml has the same volume as $1\ cm^3$ and 1000 litres is the same as $1\ m^3$. The relationships between these different units are shown in the conversion diagram below.

Can you explain how to get this diagram?



Suppose you have a box which measures 6 cm by 5 cm by 4 cm. What is its volume? Since all the dimensions are given in centimetres, you can measure the volume in cubic centimetres. Imagine filling the box with cubes of this size. six rows with five cubes in each row would cover the bottom of the box and the box would be filled by four of these layers.

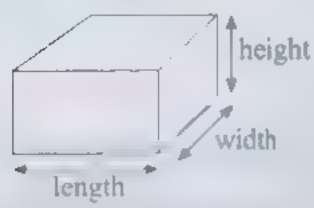
So, the total number of cubes used is $6 \times 5 \times 4 = 120$.

The volume of the box is therefore 120 cm^3.

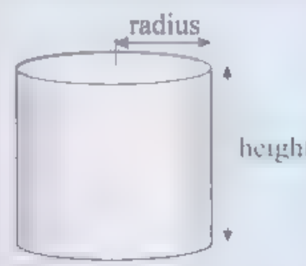
You can also use a formula to work out the volume of some of the more common solids. These are shown in the box below.

Volume formulas

Volume of a rectangular box = length × width × height.



Volume of a cylinder = area of base × height = $\pi \times (\text{radius})^2 \times$ height.



Volume of a prism = area of base × height.

a prism has a uniform cross-section

Volume of a sphere = $\frac{4}{3} \times \pi \times (\text{radius})^3$.

Activity 110 The petrol tanker

(a) The container on a petrol tanker is approximately cylindrical with an internal diameter of about 2.25 m and an internal length of 10 m. Roughly how many litres of petrol could it contain?

(b) The petrol station has an underground tank that measures 2.5 m by 4 m by 3 m. If the tank is empty, will it hold all the petrol in the tanker?

Comment

(a) If the diameter is 2.25 m, the radius will be 2.25 m ÷ 2 = 1.125 m.

The length is 10 m.

Substituting these values into the formula for the volume of a cylinder:

volume of cylinder = $\pi \times (\text{radius})^2 \times \text{height} = \pi \times (1.125 \text{ m})^2 \times 10$ m.

Rounding to 1 s.f. (or the nearest cubic metre), the volume is 40 m^3.

Since there are 1000 litres in 1 m^3, if the tanker is full, it will hold approximately 40 000 litres.

(b) The volume of a box = length × width × height. So the volume of the underground tank = 2.5 m × 4 m × 3 m = 30 m^3. This is the same as 30 000 litres and, as this is smaller than the capacity of the tanker, not all the petrol from the tanker would fit in the storage tank.

5.7 Thinking about your learning

People do learn in very different ways and if you are aware of what works well for you, it can help you to become a more effective learner. For example, in this chapter you have met a lot of new terminology and notation. How did you cope with it? Did you add these to your dictionary, make a list, draw a spray diagram, or perhaps do something else? Some people prefer to describe things in words and others find drawing diagrams more effective. Practising using new notation and terms in your writing, in discussions with others and in everyday life can also help you to feel familiar with the ideas, as can trying a few more examples on a regular basis. Making up your own examples can improve your understanding too. You may already have tried making up some scientific notation examples that you can check on your calculator or perhaps some area and perimeter puzzles using your tangram.

This chapter has concentrated on using some practical activities to help you understand ideas. Did this help you or would you have preferred a more theoretical approach? Thinking about your learning may suggest ideas for your note taking, how you work on the activities in this book and your approach to problems in the future.

As you are now over two-thirds through the course, it is worth thinking how new ideas that you meet fit in with what you already know. In this chapter, you saw how thinking back to your work on decimals helped you to discover a short cut for dealing with numbers in scientific notation and also how some of the problem strategies from Chapter 3 could be used for practical problems such as working out the area of a lawn.

Finally, do check through the study checklist to assess your progress on this chapter. Which of these topics do you think you could explain to a new student on this course? If there is an area that you are unsure about, what do you plan to do about it? Would it be better to try and sort it out soon, perhaps with help from your tutor, or would you prefer to return to it later?

Study checklist

You should now be able to:

- use your calculator for scientific notation
- understand how to carry out calculations involving powers of numbers
- interpret and draw scale diagrams
- understand some geometrical terms and notation
- work out the perimeter and area of simple shapes either by estimation or using a formula
- work out the volume of simple solids using a formula
- appreciate that drawing diagrams or working with models can help solve problems
- cope with new technical vocabulary.

6 Working with data

Numbers, words and images are examples of data.

The next census in the UK will be carried out in 2011.

In every aspect of our lives, data is collected, recorded, analysed, interpreted and used. We are bombarded with statistics too – everything from graphs of the latest sales figures to official statistics such as the census results, the inflation rate or the unemployment rate. Some of these statistics (for example, from the census) are then used in planning future provision of health services, schools and roads.

Being able to make sense of data, by recognising what is important and what is not, is therefore an important skill. This chapter considers how to summarise a set of data by using an average value and how to present data in tables, graphs and charts to make the data easier to understand. You will see how graphs can be used to monitor systems and identify trends and also how important it is to interpret the data presented critically. Graphs and charts are very useful for displaying the relationship between quantities quickly and in an accessible form, but these can also be used to mislead the unwary reader!

We return to the Sussex Ouse Conservation Society to find out more about how they use mathematics to support their case for preserving the River Ouse. This includes summarising their data by using averages and presenting their data in graphs and charts. We hope that you will then be able to apply these skills to the data, graphs and charts that you meet at home, at work or elsewhere.

6.1 Averages and spread

Think of a journey that either you or a friend does often, for example shopping for food or travelling to work. How long would you say that journey takes?

You may have said something like 'Well, it usually takes about 45 minutes to get to work, but, if the traffic is heavy, it can take up to $1\frac{1}{2}$ hours'. In other words, 45 minutes is a fairly typical time, though there can be exceptions. Now, how did you decide what time to say? Did you pick a time that seemed to occur quite often or perhaps a time that was somewhere in the middle of the times taken for recent journeys? Knowing a typical value for a data set, and also how the data is spread out around this typical value, is important for all sorts of situations. It can help you to summarise a situation and make plans accordingly, even to decide what time you might need to get out of bed!

A value which is typical of the values in a set of data is known as an **average** and it can be calculated in different ways. In this section, we look at two averages that are commonly used, the **mean** and the **median**. Usually when people talk about an 'average' they are referring to the mean value. However this is not always the case, so it is important to check what is meant by the term 'average' if you see it used in an article or report.

Calculating the mean of a set of data

The times for my journey to work during one week last month are shown in the table below.

Day of week	Mon	Tues	Wed	Thurs	Fri
Time in mins	42	58	45	47	52

(Source: Personal records)

Now, just to get a feel for the problem and to help you check that your answer is reasonable later on, what would you say is a typical length of time for the journey from this set of data? Make a note of your answer. 50

As the mean value is the most commonly used average, we will calculate this measure first.

To calculate the mean value, add all the data values together and then divide this sum by the number of values.

In this example, the sum of the values is $42 + 58 + 45 + 47 + 52 = 244$ mins.

There are five data values.
So the mean value is $\frac{244 \text{ mins}}{5} = 48.8$ mins.

Hence, the mean journey time over this week is 49 mins (to the nearest whole number).

When you have calculated the mean, check that it is somewhere between the smallest and largest value in the data set and that it is roughly what you expected. If it is not, it is worth checking your calculation. This technique can be summarised, as shown below.

Calculating a mean

- Add all the values together to find their sum
- Count the number of values
- Divide the sum by the number of values
- Write down the conclusion and include the units.

Alternatively, mean value $= \frac{\text{sum of values}}{\text{number of values}}$.

You may like to remind yourself of the society's projects by looking back at the notes you made in Activity 4 'Views on mathematics'.

In Chapter 1, you discovered that one of the aims of the Sussex Ouse Conservation Society (SOCS) is to monitor the pollution levels in the River Ouse and its tributaries to see if the river complies with the EC legislation known as the Freshwater Fish Directive (FFD). To do this, samples of water are taken at different locations at monthly intervals and the concentration of various pollutants measured. As the levels of these pollutants can vary greatly

over the year, it is quite difficult to pick one of the collected values that can be thought of as typical of the pollutant level in the river at that location throughout the year. To overcome this problem, the mean value for each pollutant at the different locations is calculated and compared with the guidelines. The FFD sets mandatory levels and more stringent guideline levels. SOCS has set its own threshold levels which are felt to be appropriate for the River Ouse system. These levels are less stringent than the FFD guideline values but more stringent than the mandatory values. Activity 111 asks you to use some of the Society's data to check whether the water at Clapper's Bridge on the Bevern Stream had an acceptable level of phosphate, which is one of the main pollutants. The amount of pollutant in a water sample is measured in milligrams per litre or mg/l. For example, if the phosphate level of a sample is measured as 3.8 mg/l, this means that overall there is 3.8 mg of phosphate in each litre of water. The mean value over a 12-month period must be less than 4 mg/l, in order for the SOCS threshold to be met.

Activity 111 Too much phosphate in the river?

The phosphate data for the water samples taken at Clapper's Bridge on the Bevern Stream between November 2004 and October 2005 is shown below.

Month	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct
Phosphate in mg/l	2.20	4.00	1.50	1.98	1.00	1.50	3.80	5.70	3.40	3.58	5.30	8.10

Source: SOCS monthly water quality reports

(a) Calculate the mean phosphate level between November 2004 and October 2005.

(b) Was the mean less than the SOCS threshold of 4 mg/l?

Comment

(a) The mean value in mg/l is:

$$\frac{2.2+4+1.5+1.98+1+1.5+3.8+5.7+3.4+3.58+5.3+8.1}{12}=\frac{42.06}{12}=3.505.$$

Hence the mean phosphate level at Clapper's Bridge over the 12-month period was 3.51 mg/l (to 3 s.f.).

(b) This level is less than the threshold value of 4 mg/l, so the water at Clapper's Bridge did meet the phosphate guidelines between November 2004 and October 2005.

Although we have calculated the mean by hand here, this kind of calculation can be carried out very quickly using a spreadsheet on a computer. SOCS monitors the levels of three different chemicals at many different locations and the mean values for the year are calculated every month. So the website designer, Mark Davis, uses a spreadsheet both to calculate the mean values and to draw the graphs.

You can also use your calculator to enter the set of data and then calculate the mean directly using one of the calculator functions.

Work through Chapter 6 of the Calculator booklet now.

Finding the median

One of the advantages of using the mean as an average value is that it does take account of all the values in the data set. However this also means that if one of the values is much higher or lower than the other values in the data set, then this can affect the value of the mean greatly. For example, suppose that there was major pollution incident in November 2005, and the recorded value for the phosphate level at Clapper's Bridge was 150 mg l. The mean for the period between December 2004 and November 2005 using this new value would be 15.8 mg l, which is higher than all the other monthly readings and substantially above the threshold value. An event like this could have a very serious effect on the wildlife in the river so, in this case, it is useful that extreme values are included. However there are situations where including extreme values may not be appropriate. For example, suppose you were calculating your average time to travel to work and, on one day, a railway strike meant that it took you six hours to get to work rather than the usual 45 minutes. Including this high value may result in a mean value that is not typical of the journey times. In such cases, you may prefer to calculate a second type of average known as the median. The median is also easier to calculate than the mean! The method for calculating the median is shown below.

Calculating a median value

- Arrange the values in ascending or descending order.
- Choose either the middle value (if there are an odd number of values) or the mean of the two middle values (if there are an even number of values).

  Write down the value of the median, including the units.

To see how to calculate a median, arrange the phosphate data values from Activity 111 in ascending order, as follows.

1.00 1.50 1.50 1.98 2.20 3.40 3.58 3.80 4.00 5.30 5.70 8.10

This is the middle of the data set
there are six values on either side

There are 12 values altogether, so there are two middle values, the sixth and seventh values.

The mean of these two middle values is $\frac{3.40 + 3.58}{2} = \frac{6.98}{2} = 3.49$.

Hence, the median phosphate value between November 2004 and October 2005 was 3.49 mg/l.

Activity 112 Calculating a median value

Calculate the median journey time using the data in the table below.

Journey					
Day of week	Mon	Tues	Wed	Thurs	Fri
Time in mins	42	58	45	47	52

(Source: Personal records)

Why is it important to put the values in order first?

Comment

Arranging the times (in minutes) in ascending order gives the following.

42 45 47 52 58

This is the middle
value in the data set

There are five values altogether, so the middle value is the third value which is 47. Hence the median journey time is 47 minutes.

Remember to include [illegible]

The middle value depends on the order in which the values have been written down. So if the values were not put in order, the middle value could be any of the values including the highest or lowest value, and these values would probably not be typical of the data set.

Activity 113 Comparing the mean and the median.

The table below shows the levels of phosphate in the water on the Bevern Stream at Streat Lane, between November 2004 and October 2005. Calculate the median value and the mean value for this data set and comment on your results.

Month	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct
Phosphate in mg/l	3.20	4.90	1.86	3.18	1.00	3.00	5.20	12.00	3.10	3.26	15.50	14.80

Source: SOCS monthly water quality reports

Comment

The mean value in mg/l (to 3 s.f.) is

$$\frac{3.2+4.9+1.86+3.18+1+3+5.2+12+3.1+3.26+15.5+14.8}{12}=\frac{71}{12}=5.92 \text{ (to 3 s.f.)}.$$

Arranging the data values in ascending order gives the following.

1 1.86 3 3.1 3.18 3.2 3.26 4.9 5.2 12 14.8 15.5

The middle two values are 3.2 and 3.26, so the median value (in mg/l) is $\frac{3.2+3.26}{2}=\frac{6.46}{2}=3.23$, hence, in this case, the median value is 3.23 mg/l and the mean value is 5.92 mg/l. The mean is much higher than the median due to the three high values of 12 mg/l, 14.8 mg/l and 15.5 mg/l. As the mean value is greater than 4 mg/l, the SOCS threshold for the phosphate level at Streat Lane has been exceeded.

Although the mean is the most commonly used average, it does depend on all the values in the data set. So if there are a few very high or very low numbers in the data set, the mean may not be a typical value in the set. The median may give a more typical measure. This also means that you need to be wary if you read an article that just talks about the 'average' value. Is it possible that choosing the mean in favour of the median (or vice versa) would strengthen the author's case?

For example, suppose a group of five employees in a small company each receive a wage of £400 a week and the director receives £1000 a week.

The mean wage is $\frac{5\times £400+£1000}{6}=\frac{£3000}{6}=£500$.

However, the median value is £400. So if there were a dispute about the level of the employees' pay, their spokesman could say, 'The average pay in this company is £400 per week', whereas the director might say, 'The average pay is £500 per week'. Both statements are technically correct because neither person has stated what kind of average they have used. However both parties

have chosen the average that best suits their cause. Which average do you think should be used here?

In situations where the data is clustered around the mean value, without any extremely high or low values, the mean and the median are likely to be similar. So comparing the mean and median values can give a clue about how the data values are spread out. However, there are some more formal measures of spread as well, as you will see in the next section.

Measuring the spread of the data

When you were trying to describe a typical time either for one of your own journeys or for our journeys, you may have stated a typical value such as 50 minutes or you may have said something like 'Well it usually takes between about 25 and 35 minutes'. In other words, you may have described how the data was spread out rather than giving a typical value. If you are trying to describe a set of data, a typical or average value is important, but it is often just as important to describe how the data values are spread out around this typical value. For example, in the phosphate data in Activity 113, the lowest value was 1.00 mg/l in March and the highest value was 15.5 mg/l in September, which is well above the SOCS threshold value and could indicate some problem with the water quality.

The difference between the greatest and least value is known as the **range** of the data set:

range = greatest value – least value.

So here the range of values at Streat Lane is
15.50 mg/l – 1.00 mg/l = 14.5 mg/l (to 1 d.p.).

In this case, the range shows that there was a large variation in the individual readings from month to month.

The range does give some idea of the spread of the data, although it is affected by either very high or very low values. There are other ways of measuring the spread of the data set that you will meet if you study mathematics or statistics further.

Using the data in Activity 111, calculate the range of the phosphate levels at Clapper's Bridge. How does this range compare with the range of levels at Streat Lane?

Comment

The highest value at Clapper's Bridge was 8.10 mg/l in October and the lowest value was 1.00 mg/l in March.

So the range of the values is 8.10mg/l – 1.00 mg/l = 7.1 mg/l (to 1 d.p). This range is less than the range of the phosphate levels at Streat Lane, so there was less variation in the results from Clapper's Bridge than in the results from Streat Lane.

This section has emphasised how you can describe a set of data by looking at an average value and how the data values are spread out around that value. Averages are used frequently in reports and in the media to summarise data. When you come across an average in this way, it's worth asking yourself some questions before drawing any conclusions.

- Do you know what kind of average has been used?
- How many values were used to calculate the average?
- How was the data collected?

The next few sections consider other ways of describing a set of data – either by compiling the results in a table or displaying them in a graph or chart.

6.2 Tables

You have already seen how tables can be used to display data clearly. Tables tend to be used when it is important that the data is displayed fairly accurately such as on a food label, or when a lot of data needs to be displayed in a concise form, such as on a bus or rail timetable. This section considers how to extract information from some more complicated tables than you have met so far in the course.

The label below is from a packet of biscuits. For health reasons, I wanted to check whether the fat content was less than 30 per cent.

NUTRITIONAL INFORMATION		
Average values	Per biscuit	Per 100 g
Energy (kJ) (kcal)	292 70	1973 470
Protein	1.1 g	7.2 g
Carbohydrate of which sugars	9.3 g 2.5 g	62.7 g 16.6 g
Fat of which saturated	3.2 g 1.5 g	21.5 g 10.0 g
Fibre	0.5 g	3.6 g
Sodium	0.1 g	0.6 g

The first thing to do is to **look at the title**. This should explain clearly what the table contains. If you are looking for a particular bit of information from a range of tables, skim-reading through the titles should enable you to find the table with the information you want fairly quickly.

In fact here there is just one table and the title is clear – the table contains the nutritional information for the biscuits.

Next, **check the source** of the data. Has the data been compiled by a reputable organisation? Can you check how the data has been collected? This is an important step; if, for example, the data is only based on a few results or has not been collected properly, you may not wish to rely on the values given in the table.

In this example, no source was given although the contact information for the consumer section of the company producing the biscuits was provided on the wrapper, and food labelling is governed by UK laws. So we can be fairly certain that this is reliable data.

Next, **examine the column and row headings**. What information is being given – raw numbers or percentages? What units are being used? Do you understand any abbreviations used and how the table is constructed?

Why do you think both these columns are included?

The two important column headings are the 'Per biscuit' and 'Per 100 g'. These columns give the values for one biscuit and for 100 g of biscuits. The row headings give the different components of the biscuit such as fat and protein. Note that the energy values are given in two different units: kilojoules (kJ) and kilocalories (kcal). The quantities of the components are given in grams (g). The units are often included in the headings, although not always, as in this case. Notice that if you add together the different components for 100 g of biscuits the total mass is 95.6 g, so some components may not have been included.

Finally, you can **look at the main body** of the table to find the information you need.

I wanted to find the fat content and check that it was less than 30 per cent. As 30 per cent of 100 g is 30 g, I need to check that the fat in 100 g of biscuits is less than 30 g. So, going down the column labelled 'Per 100 g' and across the row labelled 'Fat', the amount of fat is 21.5 g. This is less than 30 g, and so the fat content is less than 30 per cent.

Activity 115 What's in a biscuit?

How much carbohydrate is there in one biscuit and in 100 g of biscuits? What percentage of the mass of a biscuit is carbohydrate?

Comment

Going across the row labelled 'Carbohydrate' and down the column labelled 'Per biscuit', the amount of carbohydrate in one biscuit is 9.3 g.

Now looking along the same row but down the per 100 g column, the amount of carbohydrate in 100 g is 62.7 g. This is equivalent to 62.7 per cent, so approximately 63 per cent of each biscuit is carbohydrate.

The box below summarises the steps in reading a table.

Reading a table

- Read the title.
- Check the source of the data.
- Examine the column and row headings, particularly the units used.
- Check that you understand the meaning of any symbols or abbreviations.
- Extract the information you need from the main body of the table.

Activity 116 Employment in the Irish tourist industry

The table below is from the Irish Tourism Fact Card for 2003. Following the steps in the 'Reading a table' box, check that you understand the table and then answer the following questions. Note that the final column gives the percentage increase or decrease in the numbers between 2001 and 2003.

(a) How many people were employed in guesthouses in Ireland in 2003?

(b) Why do you think that there is no total for 2002?

(c) Where were most people employed in 2001?

(d) Which type of premises has shown the largest percentage rise between 2001 and 2003? How has this percentage been calculated?

(e) What do the minus signs in the final column tell you?

[illegible]

Sector/Year	1999	2001	2002	2003	%+/-('01-'03)
Hotels	53,906	54,275	54,656	54,164	–0.2%
Guesthouses	3,115	2,943	2,914	2,879	–2.2%
Self-catering accommodation	4,580	3,830	n/a	3,878	1.3%
Restaurant	40,283	41,827	41,409	41,085	–1.8%
Non-licensed restaurant	15,221	13,849	n/a	15,642	12.9%
Licensed premises	78,300	78,225	80,121	79,319	1.4%
Tourism services and attractions	33,910	34,568	34,852	34,749	0.5%
Total	229,315	229,517		231,716	0.96%

Note: Percentages rounded

Source: Fáilte Ireland's Business and Employment Survey (2004)

Comment

(a) Going across the row marked 'Guesthouses' and down the row headed 2003 shows that 2879 people were employed in guesthouses in 2003.

(b) The numbers of people working in self-catering accommodation and non-licensed restaurants were not available (as shown by the n/a entries) so the overall total cannot be found.

(c) Going down the 2001 column, the highest number in this column (apart from the total) is 78 225. So most people were employed in licensed premises.

(d) Going down the column marked '% + ('01-'03)', the highest percentage is 12.9%, which corresponds to the percentage increase in the number of people working in non-licensed restaurants.

The increase between 2001 and 2003 is 15 642 − 13 849 = 1793.

So the percentage increase is

$$\frac{\text{actual increase}}{\text{original number}} \times 100\% = \frac{1793}{13849} \times 100\% = 12.9\% \text{ (3 s.f.)}.$$

(e) The minus signs indicate that there has been a drop in the number of people employed in that sector between 2001 and 2003. The 2003 number is lower than the 2001 number.

Some tables include very large or very small numbers, such as the number of people attending a sports or arts event or the size of different pollen grains. Suppose the number of unemployed people (rounded to the nearest 1000) in four quarterly periods were 365 000, 284 000, 320 000 and 355 000.

These numbers can be expressed as 365 × 1000, 284 × 1000, and so on. So, using units of 1000, the numbers in the table can be written simply as 365, 284, 320 and 355 and the column heading changed to 'Numbers (000s)'. The (000s) shows that the numbers in the column are measured in thousands and the 365 in the column represents 365 × 1000 or 365 000.

Activity 11.2 Where did the tourists go?

This activity refers to the table below.

(a) From the title and the headings, briefly describe the data that the table contains.

(b) How reliable do you think this data is? Do you know how the data has been collected?

(c) How many domestic tourists visited Shannon in 2003?

(d) Roughly how much did all the tourists spend in Dublin in 2003?

(e) In which regions were there more domestic tourists than overseas tourists?

(f) Why do you think this data has been collected?

Euro m (or €m) is an abbreviation for 'million euros'.

What other information could you get from this table?

Regional numbers (000s) and revenue (Euro m) 2003

Numbers (000s) / ***Revenue (€m)***	**Overseas tourists**	**Nth. Ireland**	**Domestic**	**Total**
Dublin	3,444 *1,051.3*	171 *55.1*	893 *113.8*	4,508 *1,220.2*
Midlands/East	775 *310.2*	42 *9.2*	802 *90.6*	1,619 *410.0*
South-East	907 *268.4*	9 *6.6*	1,042 *138.1*	1,958 *413.1*
South-West	1,515 *619.3*	47 *16.7*	1,287 *232.2*	2,849 *868.2*
Shannon	983 *334.2*	26 *4.8*	818 *115.0*	1,827 *454.0*
West	1,159 *456.5*	82 *30.0*	1,249 *204.3*	2,490 *690.8*
North-West	476 *187.9*	219 *56.1*	566 *76.9*	1,261 *320.9*
Total revenue	*3,227.7*	*178.5*	*970.9*	*4,377.1*

Comment

(a) From the title, the table shows where tourists went in Ireland and how much they spent. Examining the headings shows that seven regions are considered and domestic, Northern Irish and overseas tourists are separated out. Both the numbers of tourists and the amount they have spent in euros (shown in the second row for each region in italics) have been recorded.

(b) No source was given on the table. However on the back of the Tourism Fact Card was the following statement.

> All estimates are based on information from the CSO's Country of Residence Survey (CRS), Fáilte Ireland's Survey of Overseas Travellers (SOT) and the Northern Ireland Passenger Survey.

CSO is the Central Statistical Office which is the government body responsible for compiling official statistics in Ireland. From the tourist card, it is not completely clear how the data was collected or estimated, although some contact details for further information were provided on the card. Counting tourists could be quite difficult – are the numbers based on people staying overnight or the number of visitors to attractions or people entering Ireland or some other way? The numbers have been rounded to the nearest 1000 and the amount of money spent to the nearest 100 000 euros.

(c) 818 000 domestic tourists visited Shannon in 2003.

(d) 1220.2 million euros (or 1 220 200 000 euros) were spent in Dublin in 2003.

(e) Comparing the columns for the overseas tourists and the domestic tourists shows that there were more domestic tourists than overseas tourists in the Midlands/East, South East, West and North West regions.

(f) The data was probably collected or estimated so that tourism initiatives (such as targeted marketing) could be planned.

Constructing your own tables

Sometimes, you may need to summarise a set of data and construct a table yourself. The first step is to decide how to sort the data into categories to stress the points you wish to make.

A 'datum' is one item from a set of data.

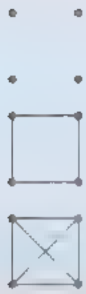
If you have a lot of data to sort, you may like to use a **tally chart**. To do this, first draw up a blank table with the row and column headings. Then taking each datum in turn, put a vertical mark in the cell to which it belongs. When you have counted the item, you may find it helpful to cross it off your list, so that you know which item you are up to – it is easy to lose track! When you get to the fifth datum in the cell, draw a diagonal line across the previous four marks, making a 'gate' like this: ~~||||~~. This makes it easier to total the number in each cell, as you can add groups of five together quickly. When you have completed the tally chart, check that the total number of tallies on your chart agrees with the number of items on your data list.

For example, suppose you have volunteered to collect the sandwiches for your colleagues at work. There is a choice of cheese (C), ham (H) or salad (S) and the order is as follows.

H H C S H C C H H C S H

The tallies for the sandwich order would look like the following.

Cheese	||||
Ham	~~||||~~ |
Salad	||



There are different ways of drawing up a tally table. John Tukey, a statistician, suggested using a square symbol, putting one dot in each corner for the first four items of data, then a side for each of the next four items and finally a diagonal line for each of the last two items. This symbol then represented 10 data items. Tukey felt that this symbol was easier to read and add up than the gate symbol. What do you think?

Activity 118 The tourist market

Imagine you are the manager of a small Irish hotel that has guests of different ages and nationalities. You would like to know what kinds of guests visit your hotel, so you decide to summarise this information in a table.

[illegible]

You decide to divide the ages into groups: child (under 16), adult (16 – 60) and senior (over 60) and the nationalities into the categories Irish (Ir), British (B), Mainland European (E) and the Rest of the World (W).

From the room bookings the following data was collected. This is of the form 16E where the number represents the age (16) and the letter represents the category (E), so 16E represents a 16-year-old guest from Mainland Europe.

16E	7B	24Ir	26Ir	46Ir	43Ir	5Ir	50W	55W	13Ir	13B	15Ir
61B	8Ir	37W	48E	8Ir	62B	49E	6W	55Ir	11Ir	9Ir	12B
62W	65B	65B	67Ir	13W	12W	54B	72Ir	61Ir	48B	61W	15B
62Ir	67E	10W	27B	12Ir	31B	35W	8W	42B	43B	15W	

Construct a blank table with a title, the source of the data and column and row headings corresponding to the categories above. Use a tally to determine the number of guests in each age group and each category in your table first. From the tally chart, construct the final table.

Comment

The final table should look something like the table below. You may have put the rows and columns the other way round.

Origin and age categories of hotel guests

	Child	Adult	Senior	Total
Irish	8	5	4	17
British	4	6	4	14
M. European	0	3	1	4
Rest of World	6	4	2	12
Total	18	18	11	47

Source: Hotel accommodation records

Note that you can work out the total number of visitors either by adding together the numbers of children, adults and seniors ($18 + 18 + 11 = 47$) or by adding the numbers of Irish, British, Mainland European and Rest of World visitors ($17 + 14 + 4 + 12 = 47$). These two totals should agree, so it is useful to work out both as a check.

Activity [illegible] Analysing the tourist table

From the table in Activity 118, work out the following.

(a) The percentage of visitors that are children.

(b) The percentage of visitors that come from outside Britain.

(c) Why is it helpful to calculate percentages here? What other calculations might be useful?

Comment

(a) Eighteen out of the 47 visitors are children. So the percentage of visitors that are children is given by:

$$\frac{\text{number of children}}{\text{number of visitors}} \times 100\% = \frac{18}{47} \times 100\% \approx 38\%.$$

(b) 14 visitors came from Britain. So the number of visitors that came from outside Britain is $47 - 14 = 33$. (Or the number of non-British visitors is $17 + 4 + 12 = 33$).

So the percentage of visitors that are not British is given by:

$$\frac{\text{number of non-British visitors}}{\text{total number of visitors}} = \frac{33}{47} \times 100\% \approx 70\%.$$

(c) Percentages indicate the proportion of guests in each category, so, for this sample of data, just over a third of the visitors were children. This sort of information may be useful for marketing or development purposes. Many other calculations can be made such as the percentage of seniors or the percentage of visitors from Ireland, and so on.

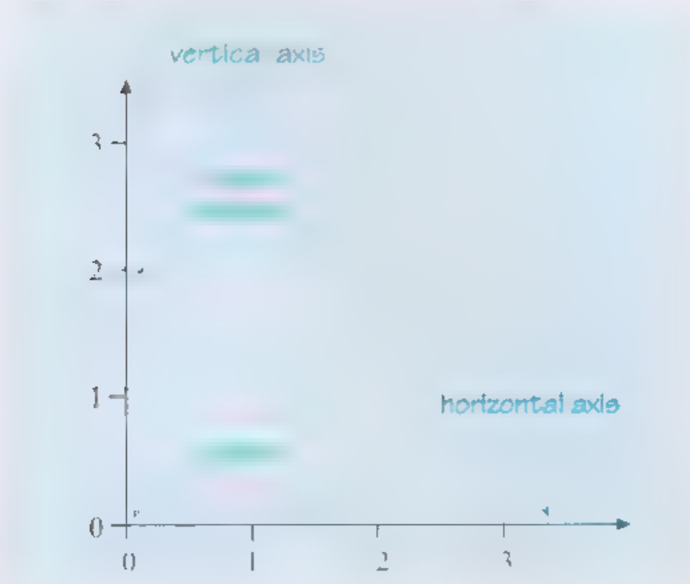
Displaying data concisely in a table is also useful if you wish to draw a graph to illustrate some features of the data, as you will see in the next section.

[illegible]

Although tables are very useful for summarising a lot of data concisely, clearly and accurately, sometimes you may wish to get an overall message across to your readers more quickly by using a graph or chart. Graphs can also be used to explore relationships between sets of data.

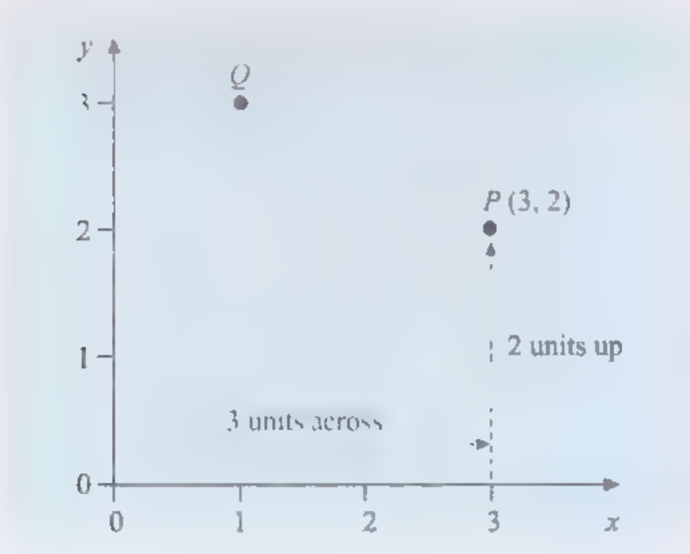
Many graphs and charts are constructed by plotting points on a grid.

The graph below shows two perpendicular lines: the **horizontal axis** and the **vertical axis**, which set up the grid. Each axis is marked with a **scale**, in this case from 0 to 3. The point where the two axes meet and where the value on both scales is zero is known as the **origin**.



Now you can describe the location of any point on the grid by its horizontal and vertical distance from the origin.

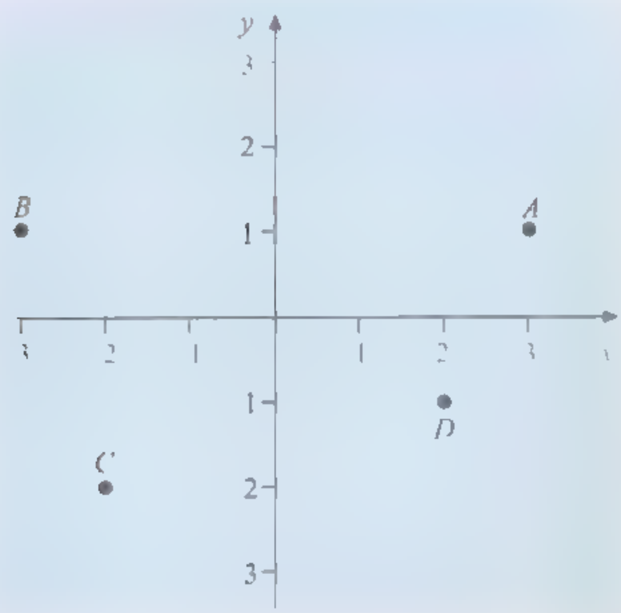
In the graph below, the horizontal axis has been labelled with an x and the vertical axis with a y. The horizontal axis is usually known as the x-axis and the vertical axis as the y-axis. To get to the point P from the origin, move 3 units across and 2 units up. (Alternatively the point (3, 2) is opposite 3 on the x-axis and 2 on the y-axis.) We say that P has the **coordinates** (3, 2). The horizontal distance of P from the origin is known as the x-coordinate of P and the vertical distance from the origin is known as the y-coordinate of P. So for point P, 3 is the x-coordinate and 2 is the y-coordinate.



In the same way, Q is 1 unit across and 3 units up, so Q has coordinates (1, 3).

Alternatively, Q is opposite 1 on the x-axis and is opposite 3 on the y-axis.

The grid above can be extended downwards and to the left to include negative values as shown below.



Here, B is 3 units to the **left** of the origin so its x-coordinate is −3. It is 1 unit up, so its y-coordinate is 1. So B has coordinates (−3, 1). (Alternatively, B is opposite −3 on the x-axis and B is opposite 1 on the y-axis.)

Activity 120 Reading coordinates

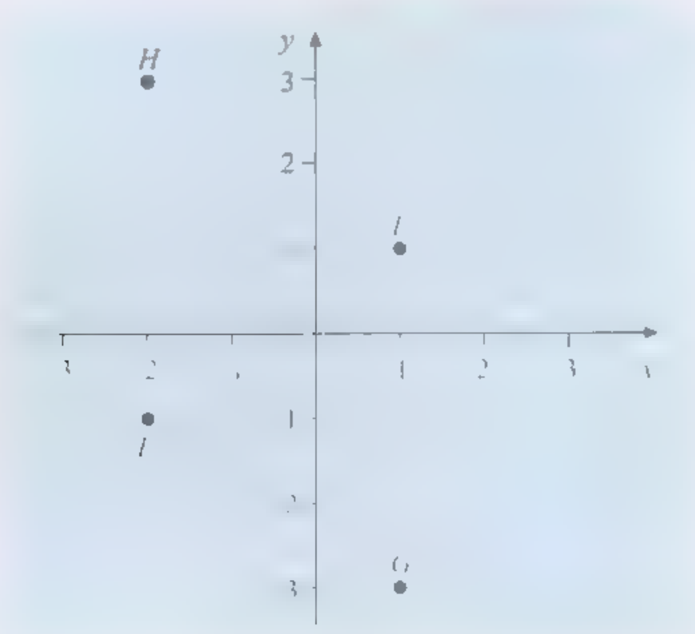
(a) Write down the coordinates of points A, C, D and the origin, shown in the graph above.

(b) On the same grid, mark the points E at (1, 1); F at (−2, −1); G at (1, 3) and H at (−2, 3). You can use a dot (•), a small circle with a dot at the centre (◉) or a small cross (×) to mark each point.

Comment

(a) A is opposite 3 on the x-axis, so its x coordinate is 3. It is opposite 1 on the y-axis, so its y-coordinate is 1. The coordinates of A are (3, 1). C is opposite −2 on the x-axis, so its x-coordinate is −2. It is opposite −2 on the y-axis, so its y-coordinate is −2. The coordinates of C are (−2, −2). Similarly, the coordinates of D are (2, −1) and the origin has coordinates (0, 0).

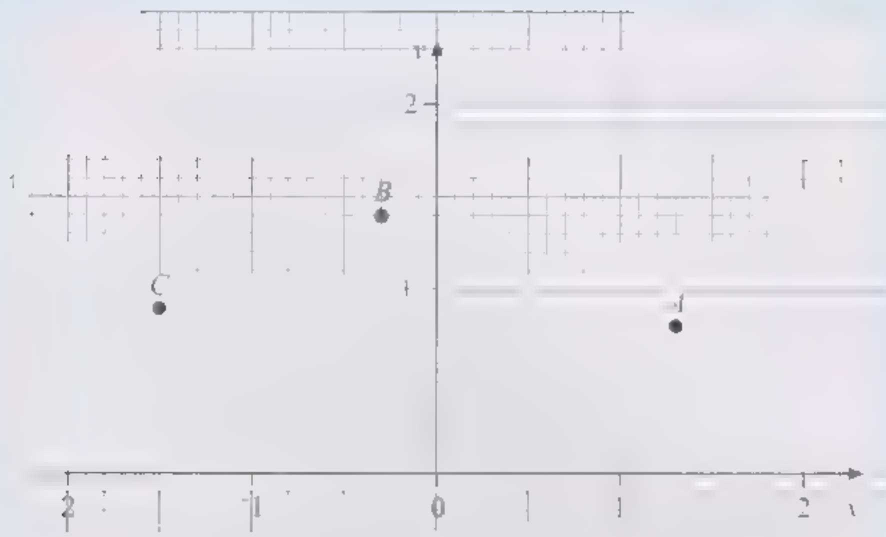
(b) E, F, G and H are shown on the grid below. E has coordinates (1,1) so is plotted 1 unit to the right and 1 unit up, that is, opposite 1 on the x-axis and opposite 1 on the y-axis. F has coordinates (2, 1), so F is plotted opposite 2 on the x-axis and opposite 1 on the y-axis. The coordinates of G are (1, 3) so G is plotted opposite 1 on the x-axis and opposite 3 on the y-axis. The coordinates of H are (2, 3) so H is plotted opposite 2 on the x-axis and opposite 3 on the y-axis.



In the examples above, all the coordinates were whole numbers and the scales were also marked at whole unit intervals. It is possible to plot points which have decimal coordinates such as (1.8, 2.1) on these grids, but it is easier if graph paper is used. The scale can then be used to work out the position of the points.

In the graph overleaf, 2 cm represents 1 unit. Each unit is represented by 10 intervals marked on the graph paper, so each interval represents 1/10 or 0.1 of a unit. The point A has coordinates (1.3, 0.8) and is plotted 1.3 units across to the right of the origin and 0.8 units up. This is 3 intervals past the 1 mark on the horizontal scale and 8 intervals up from the origin, i.e. opposite 1.3 on the x-axis and opposite 0.8 on the y-axis.

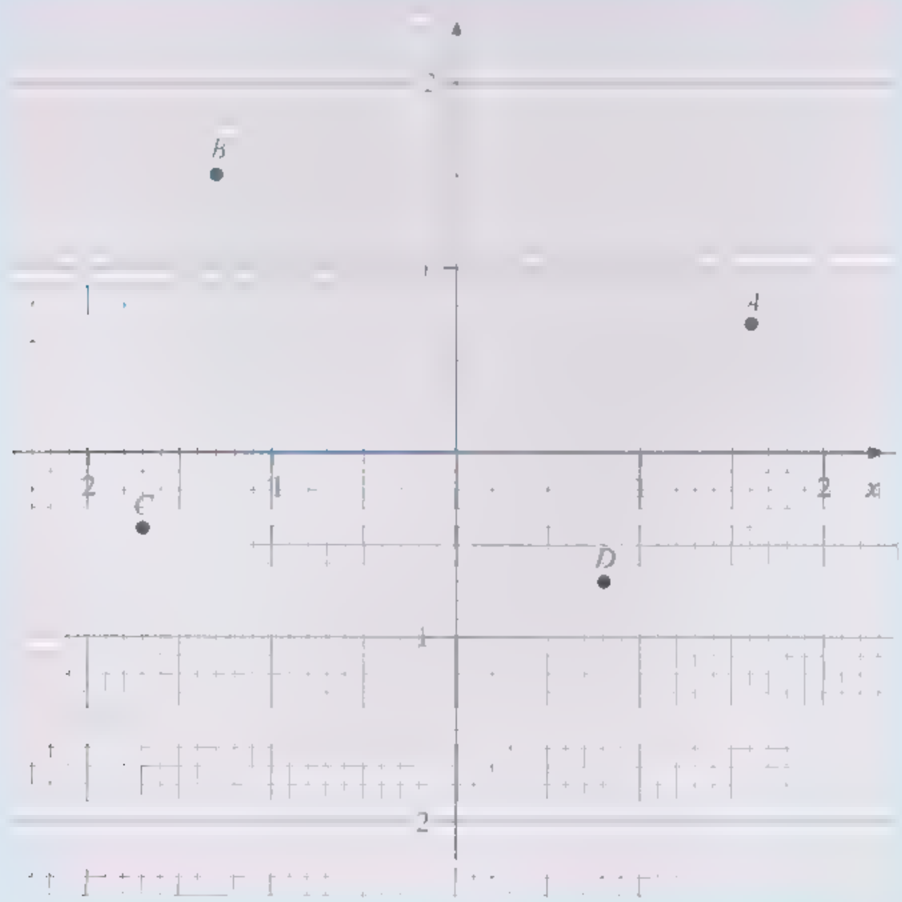
Similarly, B is plotted at the point which is 3 intervals to the left of the origin on the horizontal scale and 4 intervals past the 1 mark on the vertical scale, i.e. opposite 0.3 on the x-axis and opposite 1.4 on the y-axis. So the coordinates of B are (0.3, 1.4). Similarly, the coordinates of C are (1.5, 0.9).



Activity 121 More coordinates

Read off the coordinates of points A, B, C and D in the graph below.

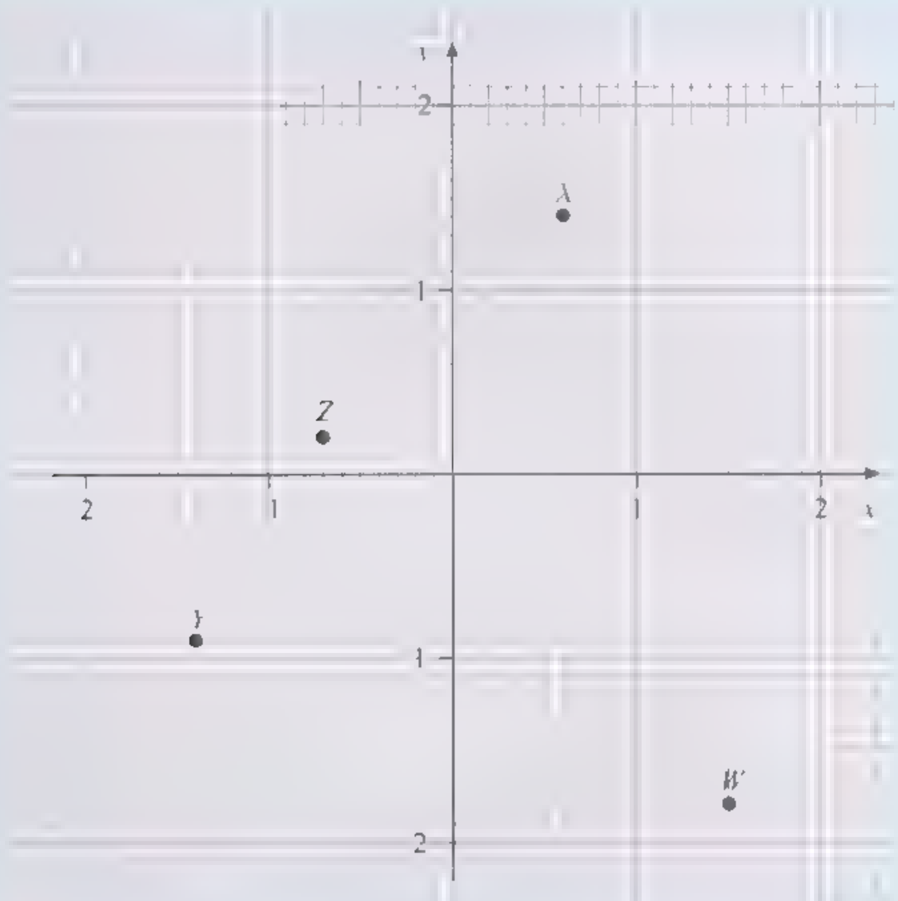
On the same grid mark the points, W at (1.5, 1.8); X at (0.6, 1.4); Y at (1.4, 0.9) and Z at (0.7, 0.2).



Comment

As each interval represents 0.1 and A is 6 small intervals past 1 horizontally, the x coordinate of A is 1.6. A is 7 intervals up, so the y coordinate is 0.7. Hence A has coordinates (1.6, 0.7). Similarly B has coordinates (1.3, 1.5), C has coordinates (1.7, ˉ0.4) and D has coordinates (0.8, 0.7).

W, X, Y and Z are shown in the graph below.



Displaying data on a graph or chart

In the examples considered so far, you have seen that, provided you have both an x- and a y-coordinate, you can plot a point on a chart. So if you have a set of data which consists of pairs of data values, these can also be plotted. For example, suppose a nurse is monitoring the mass of a woman during the last few weeks of her pregnancy and has recorded the woman's mass each week as shown in the table below.

[illegible] during pregnancy

Week of pregnancy	34	36	38	40
Mass/kg	75.6	77.4	78.1	78.5

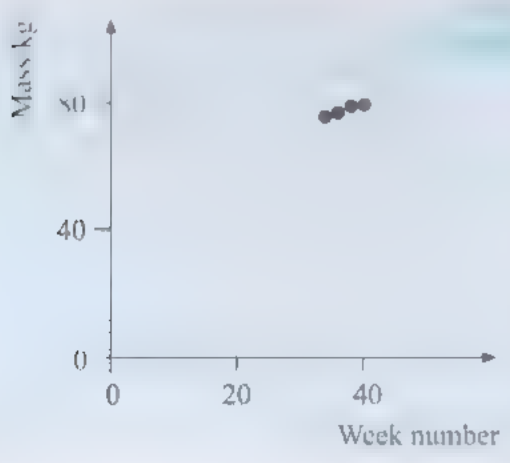
Source: Hospital records

The second row heading (Mass/kg) shows that mass is measured in kg. So in week 34, the woman had a mass of 75.6 kg. This can be written as (34, 75.6) and plotted on a graph in the same way as the earlier examples. If we take the x-coordinate of each point as the week number and the y-coordinate as the mass in kg, points can be plotted at (34, 75.6), (36, 77.4), (38, 78.1) and (40, 78.5) to represent this set of data.

Maths and science courses often use the slash notation for headings in tables and labels on graph axes. If you consider Mass/kg as a fraction and then substitute the measurement, the units cancel, leaving a pure number that is then shown in the table or on the graph scale: For example, $\frac{75.6\ \text{kg}}{\text{kg}} = 75.6$.

Choosing the scales for a graph or chart

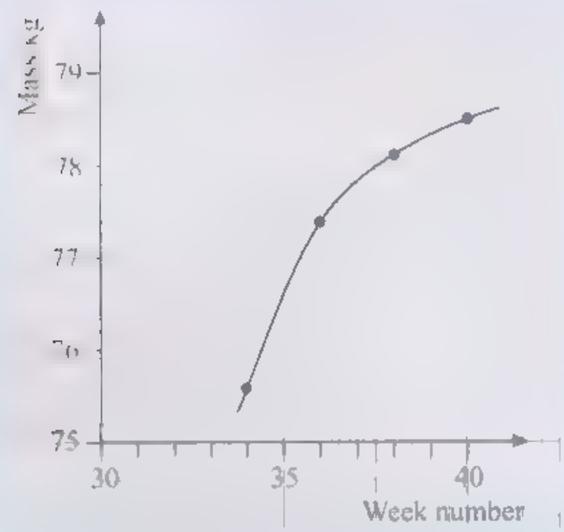
In the examples considered so far, the scales on both axes have been provided for you. The scales have been the same on both axes and they have both started at zero. However this is not always necessary. Scales are usually chosen to illustrate the data clearly, making good use of the graph paper and using a scale that is easy to interpret.



In this example, if the scales started at 0, all the points would be plotted in the top right-hand corner and would be difficult to read and distinguish from each other.

The y-values range from 75.6 to 78.5 and the x-values start at 34 and end at 40. So, the difference between the lowest and highest y-coordinate is 2.9 and the difference in the x-coordinates is 6. So a clearer way of presenting this data would be to use a scale on the x-axis which starts at 30 and ends at 40, and a scale on the y-axis which starts at 75 and ends at 80. Using these scales, the points could be plotted as follows.

Graph to show mass gain during last four weeks of pregnancy
Source: Hospital records



When the scales have been marked on the axes, remember to label the axes to show what is being measured and include the units. Then plot the points and if appropriate, join them with a line or smooth curve. In this case, you would expect the woman's mass to change steadily between hospital visits over this period, so it is reasonable to join the points with a curve to show the overall trend. Add a title so that your reader can see at a glance what your graph illustrates. Finally state the source so your reader can check the data if they wish.

Activity 12 Different scales

In the graph above, what scales have been used on the axes?

What does each small (2 mm) interval on the graph paper represent on the x- and y-axis?

Comment

On the horizontal axis, the edge of one large square (or 2 cm) represents 5 weeks. So the edge of one small square (or 2 mm) represents 5 weeks $\div$ 10 = 0.5 week, since there are 10 small squares along the edge of each large square.

On the vertical axis, the edge of each large square (or 2 cm) represents 2 kg, so the edge of each small square (or 2 mm) represents 2 kg $\div$ 10 = 0.2 kg. When you are plotting points, try to choose scales that make good use of your paper and that are easy to interpret as well – multiples of five and ten often work well.

Since you would expect the woman's mass to change steadily between hospital visits, it is possible to use the graph to estimate her mass at other times during the six weeks.

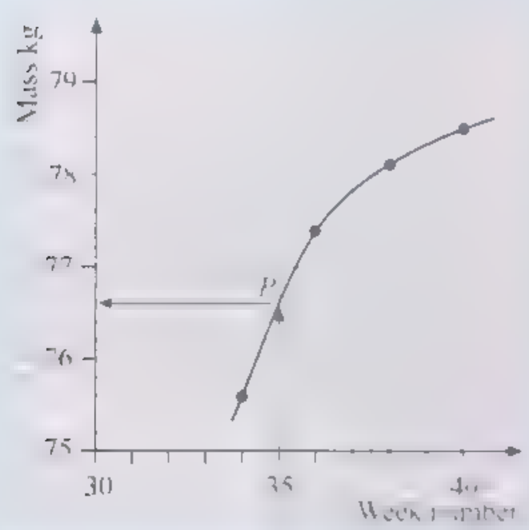
For example, to estimate her mass at 35 weeks, first find 35 on the horizontal axis.

Then from this point, draw a line parallel to the vertical axis, until the line meets the curve at point P.

From P, draw a line horizontally, to meet the vertical axis. Read off this value – here, it is 76.6.

So an estimate of the woman's mass at 35 weeks is 76.6 kg. Reading off values between the plotted points is known as **interpolation**.

Graph to show mass gain during last six weeks of pregnancy
Source: Hospital records



Activity 12 Reading values from a graph

(a) Use the graph to estimate the woman's mass at 37 weeks.

(b) When do you think the woman's mass was 76 kg?

(c) From the graph, can you predict the mass of the woman at times later than 40 weeks?

Comment

(a) Find 37 on the horizontal axis and draw a line vertically up to the curve. Then draw a line horizontally from this point to intersect the vertical axis This value is approximately 77.8. So the woman's mass at 37 weeks is estimated to be 77.8 kg.

(b) Find 76 kg on the vertical axis. Draw a horizontal line to meet the curve. Then draw a line vertically down to meet the horizontal axis. Read off the value – here it is about 34.5. So the woman's mass is estimated to be 76 kg after about 34.5 weeks.

(c) No, a woman's pregnancy usually lasts about 40 weeks. When the baby is born, there will be a sudden reduction in her mass, so it is not appropriate to use the graph to estimate her mass after 40 weeks.

Trying to estimate the coordinates of points on the curve that lie outside the plotted points is known as **extrapolation**. It can sometimes be used to find values that are close to the plotted data set, if you are confident that the graph continues in a similar manner. However, you do need to be cautious, as shown in the last activity.

The graph can also be used to determine the overall trend in the data. In this case, the graph emphasises the change in the woman's mass during the last few weeks of her pregnancy. It shows that she gained mass quite rapidly between weeks 34 and 36, but from week 36 to week 40, she gained less.

Activity 124 Drawing and interpreting a graph

In Chapter 5, Activity 100, the following data values were collected on the circumference and diameter of various circular objects.

4.4	14.1
6.6	20.9
[illegible]6	27
1[illegible]3	32.3
[illegible]	7[illegible]8

Plot these points on a graph with the diameter on the horizontal axis and the circumference on the vertical axis. Draw a straight line through the points.

Use your graph to estimate:

(a) the circumference of a circle of diameter 15 cm

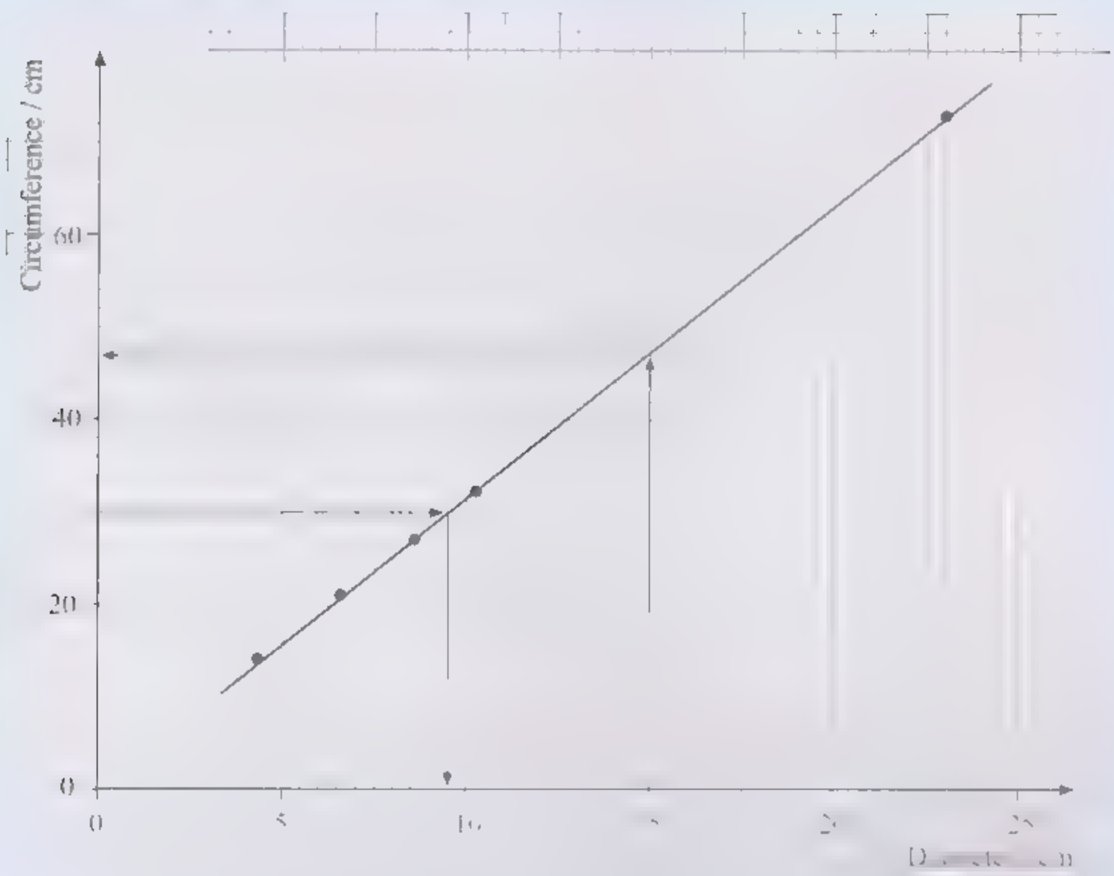
(b) the radius of a circle of circumference 30 cm.

Comment

(a) To find the circumference of a circle of diameter 15 cm, find 15 cm on the horizontal axis. From this point, draw a vertical line to meet the graph. Then draw a horizontal line across to the vertical axis. Read off the value where the line crosses the axis. Here it is about 47 cm since each small interval on the vertical axis represents 2 cm.

So the circumference of a circle of diameter 15 cm is about 47 cm.

Graph to show the circumference of household objects
Source: Own measurements



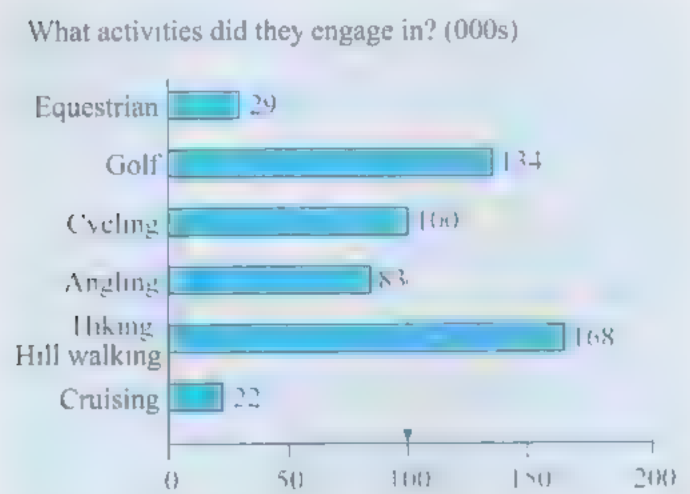
(b) If the circumference is 30 cm, find the point that represents 30 cm on the vertical axis. From this point draw a line horizontally across to the graph, then vertically down to the horizontal axis. The point where this line crosses the horizontal axis gives the diameter of a circle whose circumference is 30 cm. It is about 9.5 cm. So the radius of the circle will be half of this, about 4.7 to 4.8 cm.

Notice that in exploring the relationship between the circumference and diameter of a circle, we have now used several different approaches – a practical one that involved measuring, drawing up a table of values and looking at the ratio $\frac{\text{circumference}}{\text{diameter}}$, using a formula and now drawing a graph. Looking at a problem in different ways like this can strengthen your understanding and help you to tie ideas together. So if one way doesn't work for you, you can always try another one!

Understanding how to read off the coordinates of a point and also being able to plot points accurately will help you both to interpret and to draw a wide range of different graphs and charts.

6.4 Bar charts

Not all the information on the Irish Tourism Fact Card was presented in a tabular form. In particular, the numbers of people joining in different activities were illustrated by using a **bar chart** as shown below. (The data was based on estimates from the CSO 'Purpose of Visit' survey.)



In a bar chart, the length of each bar represents the number in that category. On this chart, you can see clearly that the longest bar is for hiking hill walking and so this was the most popular activity. The shortest bar is for cruising so this was the least popular activity of those listed.

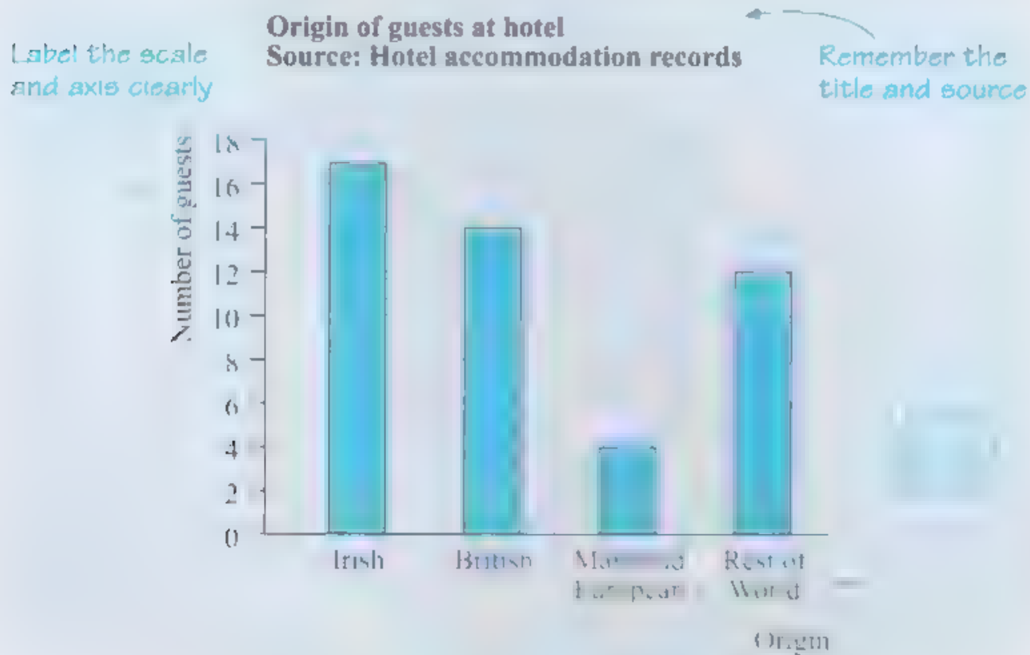
From the title, we know that the numbers of visitors are measured in thousands. On this bar chart, the values represented by the bars have been marked directly on the chart, so these can be read off straightaway. For example, the number of visitors who played golf was 134 000 and the number of visitors who went hiking hill walking was 168 000. If the values had not been marked on the bars, they could have been estimated by drawing a line from the end of the bar and seeing where it intersected the horizontal axis. For example the top of the cycling bar is level with the '100' marked on the horizontal scale (as shown by the dashed line), so approximately 100 000 visitors went cycling. Since the bars on this chart are horizontal, the chart is known as a **horizontal bar chart**. Bar charts can also be drawn with the bars vertical. Note that in a bar chart, each bar has the same width and since the bars represent different and unrelated categories, the bars do not touch each other and are separated by a gap.

Activity 12[illegible] The tourist market again

On squared paper, draw a vertical bar chart using the data from Activity 118 to show the number of tourists in each of the nationality categories. (The nationality categories should be marked on the horizontal axis and the number of visitors on the vertical axis.)

Comment

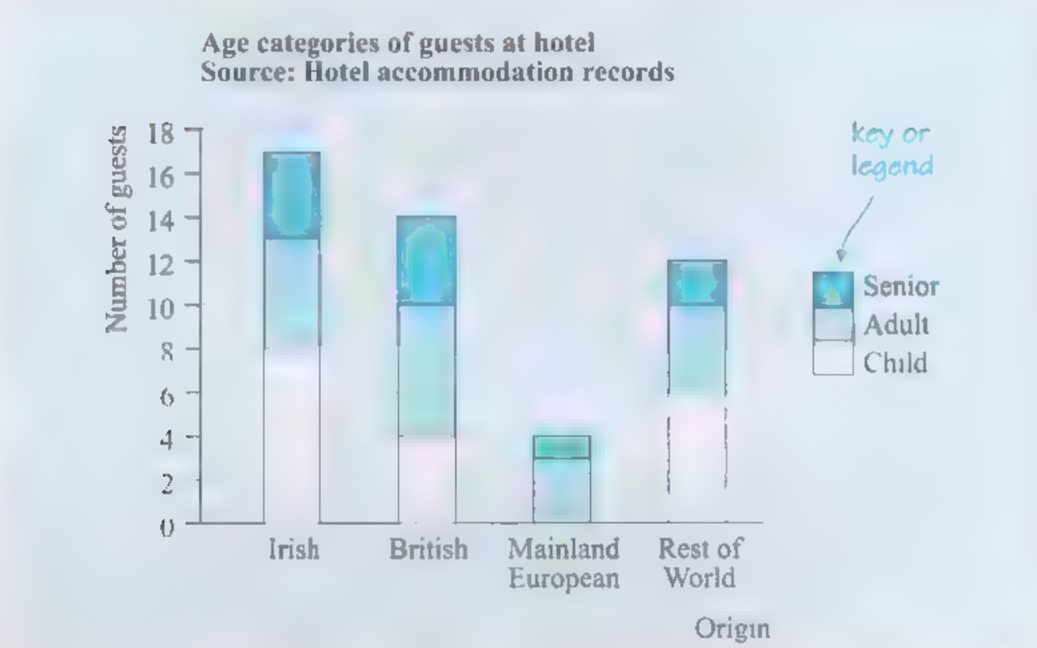
The bar chart is shown below. You can either draw the graph by hand on graph paper or if you prefer use a spreadsheet program on a computer. In either case, please make sure that you have included the title and source of the data and that you have labelled the axes and marked the scale clearly.



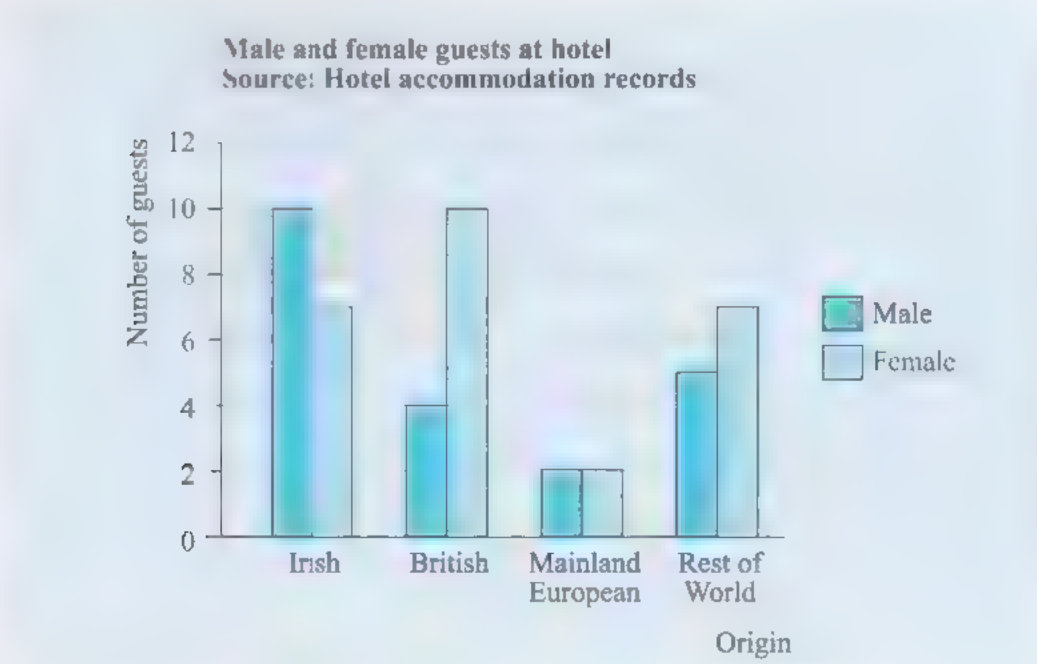
Notice how this bar chart stresses the **total** number of tourists in each category. You can see easily that there were more tourists from Ireland than any of the other groups. If you also wished to show how the totals were broken down into the different age groups, you could split each bar into three different sections where the length of each section represented the number of

guests in that age group. The resulting **component bar chart** is shown below. Notice that a **key** or **legend** has been added to the graph to explain the shading for the different categories.

When you read a component bar chart, you need to find the height of the relevant section. For instance, the number of British adults is 6. The top of that section is at 10, the bottom is at 4, so the value is 10 – 4 = 6.



Another way of displaying the totals would be to split each bar into two, one representing the number of males and the other the number of females in each category. To enable these numbers to be compared directly, the bars representing males and females can be placed next to each other as shown below.



Activity [illegible] Male and female

Using the bar chart above, answer the following questions.

(a) How many female guests from Britain were there?

(b) How many male guests stayed at the hotel during the summer season?

Comment

(a) There were 10 female guests from Britain.

(b) The total number of male guests was $10 + 4 + 2 + 5 = 21$.

Notice how all these charts followed the same format with the title and source clearly labelled and the axes and scales clearly marked, but they emphasised different aspects of the data. When you are displaying data in a graphical form, it is important to choose a chart or graph which stresses the main points as simply as possible, so that your reader can understand your chart quickly and easily.

The tourist market bar charts display **how many** items (in this case, guests) there are in each category. How many times an item occurs is known as the **frequency**. Charts that display frequencies are also known as frequency diagrams.

6.5 Pie charts

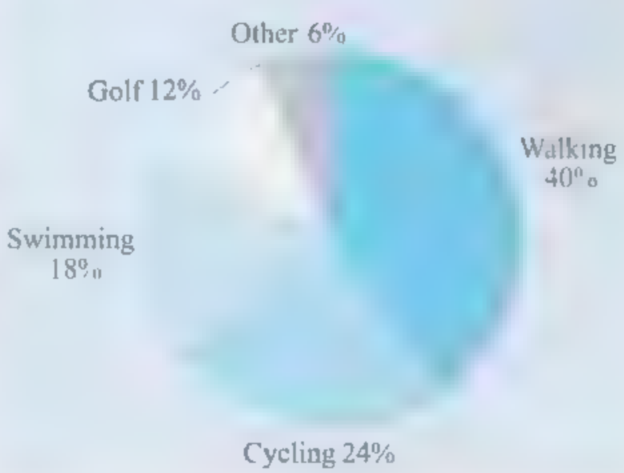
Although bar charts are useful to display numbers and percentages, sometimes you may wish to stress how different components contribute to the whole. Pie charts are often used where you want to compare different proportions in the data set. The area of each slice of the pie represents the proportion in that particular category.

> The mathematical term for a slice is a **sector**.

For example, the pie chart below illustrates the favourite type of exercise for a group of people.

> Florence Nightingale (1820–1910) was a statistician as well as a nurse – she invented 'polar area diagrams' (which are similar to pie charts) to illustrate the number of deaths of soldiers from different causes.

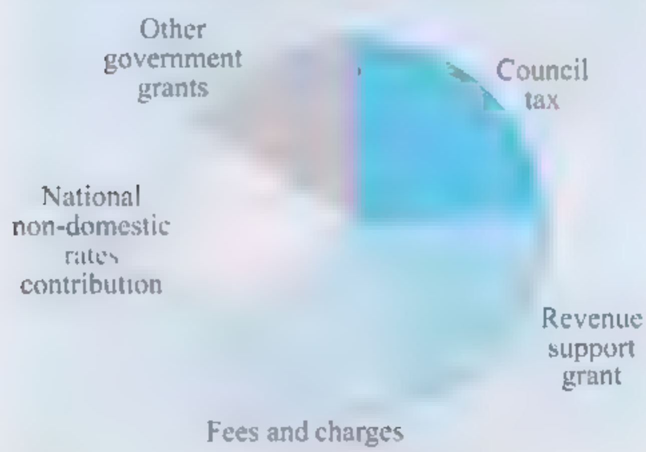
Favourite activities
Source: OU Survey 2005



This pie chart shows the percentages for each category, so you can read these off directly. The most popular activity was walking (40 per cent), followed by cycling (24 per cent) and swimming (18 per cent). Notice however that although the source of the data has been stated, we do not know how many people were interviewed. It is important to consider this. If it was a small sample, you may not wish to rely on the results to generalise to a larger group, for example.

Sometimes the percentages are not marked on the chart. The proportions can then be roughly estimated by eye. For example, the pie chart below illustrates how Worcestershire County Council's budget of £453 345 000 for 2004/05 was made up.

Worcestershire County Council Budget 2004/05
Source: WCC Statement of accounts 2004/05



The NNDR is a tax collected locally and then sent to central government, before being redistributed to councils.

The pie chart represents the different types of income received by the council. For example, the sector representing the income from council tax is just under $\frac{1}{4}$ of the pie. So, just under a quarter (or 25 per cent) of the income came from council taxes. The sectors representing the income from the revenue support grant, the national non-domestic rates contribution (NNDR) and fees and charges are all about the same size and roughly $\frac{1}{5}$ of the pie chart each. So, each of these sources of funding contributed about 20 per cent of the total amount. The smallest contribution came from other government grants.

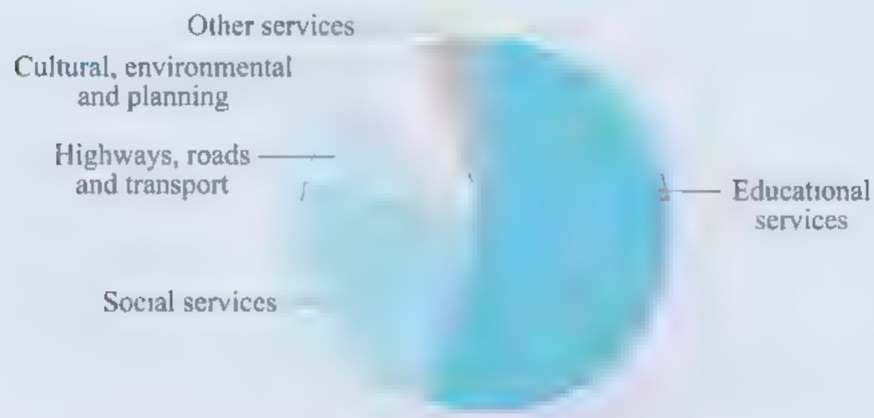
If necessary and provided the pie chart has been drawn accurately, you can also work out the percentages by either measuring the angle at the centre of the pie for each slice, using a protractor or by measuring sections of the circumference. For example, the angle at the centre of the circle for the 'Council tax' category is about 86°. Since the whole angle at the centre of the circle is 360°, the fraction is $\frac{86}{360} \approx 0.24$ or 24%. So about 24 per cent of the funding came from council taxes.

This agrees with the estimate made earlier. Pie charts are often used to give an overall impression rather than detailed information, so on many occasions a rough estimate will suffice.

Activity 12 Where did the money go?

The pie chart below illustrates how Worcestershire County Council spent its budget. What does the pie chart show?

Worcestershire County Council – gross expenditure 2004/05
Source: WCC Statement of accounts 2004/05



Comment

Over half of the expenditure was on educational services and just under a quarter of the expenditure was on social services. The remaining expenditure (in decreasing order) was on highways, roads and transport, cultural environmental and planning and other services. These fractions are quite difficult to estimate by eye. However, together these three categories accounted for under a quarter of the total expenditure. About twice as much was spent on the highways, roads and transport as on the category labelled 'Other services'.

If you have access to a computer, many spreadsheet programs provide facilities to draw pie charts from a set of data. Although these programs often offer several different ways of displaying a pie chart, a simple chart like those in this section often gets the message across simply and clearly.

If you wish to draw a pie chart by hand, you will need to work out the angle at the centre for each sector and then use either a pie chart template or a protractor to measure the angles on the chart. In the next section, we look at using charts and graphs in a practical situation.

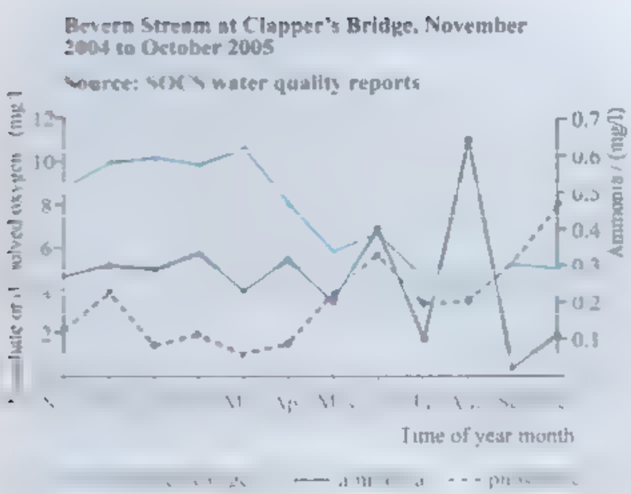
6.6 A case study

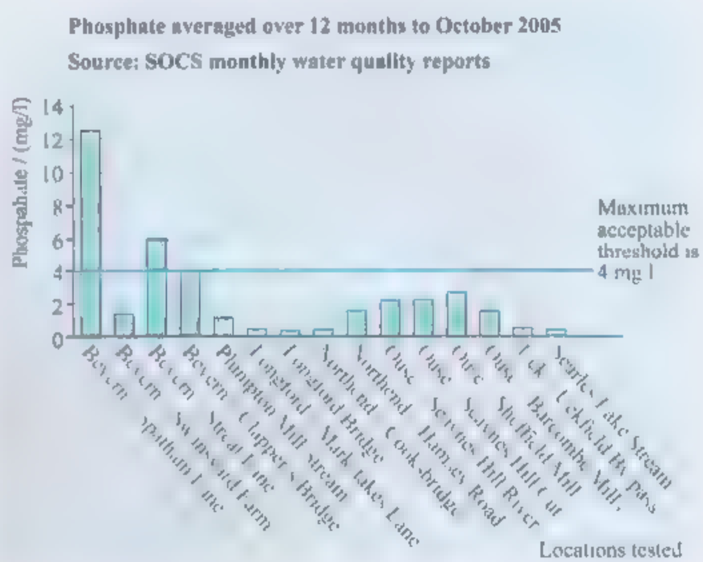
This section suggests guidelines for interpreting graphs and charts in general and develops your skills further, by considering how the Sussex Ouse Conservation Society uses different kinds of graphs on their website to inform people of the pollution levels in the River Ouse system. In the next activity, Mark Davis and Jim Smith discuss the conservation of the river and some of the factors that they considered when designing the website.

Activity 12 Designing the website

Listen to audio track 3 and make a note of the data that is collected and why Mark decided to display the data in bar charts and line graphs rather than tables.

The two charts below are from the website that Mark and Jim discuss briefly. We will consider how to interpret these graphs in detail later.





Then think about your own experience of getting information from charts and tables.

What do you think are the advantages of using a table and of using a graph or chart?

Comment

Samples are taken from different locations on the tributaries of the River Ouse every month, and the amounts of ammonia, phosphate and dissolved oxygen are measured. The data used to be summarised in tables, but by drawing bar charts and line graphs, people could quickly assess whether the levels of these different substances were within acceptable levels for the river and also

spot any trends. So, for example, if the amount of phosphate appeared to be rising in a particular area, further sampling and investigations could be carried out. If there was a pollution incident, other organisations could be informed quickly so that appropriate action could be taken. From your own experience, you may have discovered that tables are useful for detailed accurate information such as on a train timetable and graphs are useful for looking at an overall trend quickly.

Before analysing the graphs and charts in detail, try the next activity to summarise what you learned in Sections 6.3, 6.4 and 6.5.

Activity 120 How to interpret a graph or chart

Jot down your own list of tips for interpreting a graph or chart. (You may find it helpful to refer back to the 'Reading a table' box in Section 6.2.)

Comment

Reading a chart or a graph is similar to reading a table.

Reading a chart or graph

- Read the title of the chart and check the source of the data.
- Check the axes – what is being measured? What units have been used?
- Examine the scales – how should these be interpreted? What does each interval represent?
- Check the key or legend – what does the shading or line represent?
- Finally examine the chart or graph itself, reading off the information that you need. Are you looking for a specific value or a trend or some other information?

In the audio, Mark described the two different kinds of graphs that he uses to display the water quality data for the river. To compare the water quality at different locations, three bar charts are drawn which show the typical levels of ammonia, phosphate and dissolved oxygen over the past year. The bar chart for ammonia for the period November 2004 to October 2005 is shown below.

Ammonia level averaged over 12 months to October 2005
Source: SOCS monthly water quality reports

From the title, the bar chart shows the average levels of ammonia for the 12 month period up to October 2005, at the different river locations. The data has been obtained from the monthly water quality reports.

Having checked the title and the source, the next step is to look at the axes – what is measured, what units are used and how should the scales be interpreted?

On the horizontal axis, the different river locations are marked. As these are just names to identify where the samples have been taken, no scale or units are required.

Milligrams per litre can also be written as $mg\,l^{-1}$.

The amount of ammonia is measured on the vertical axis. The units used are mg/l or milligrams per litre. The scale is marked every 0.05 mg/l, starting at 0.00 mg/l.

The length of each bar represents the amount of ammonia (measured in mg) that is present in each litre of water. For example, if you draw a horizontal line along the top of the bar that represents the Bevern Stream at Spatham Lane, it crosses the vertical axis just above 0.4 mg/l.

By dividing the axis between 0.4 and 0.45 into five equal intervals, you can see that the concentration of ammonia is probably just under 0.42 mg/l.

Can you read off the concentration of ammonia at Longford Bridge? You should find that it is about 0.24 mg/l.

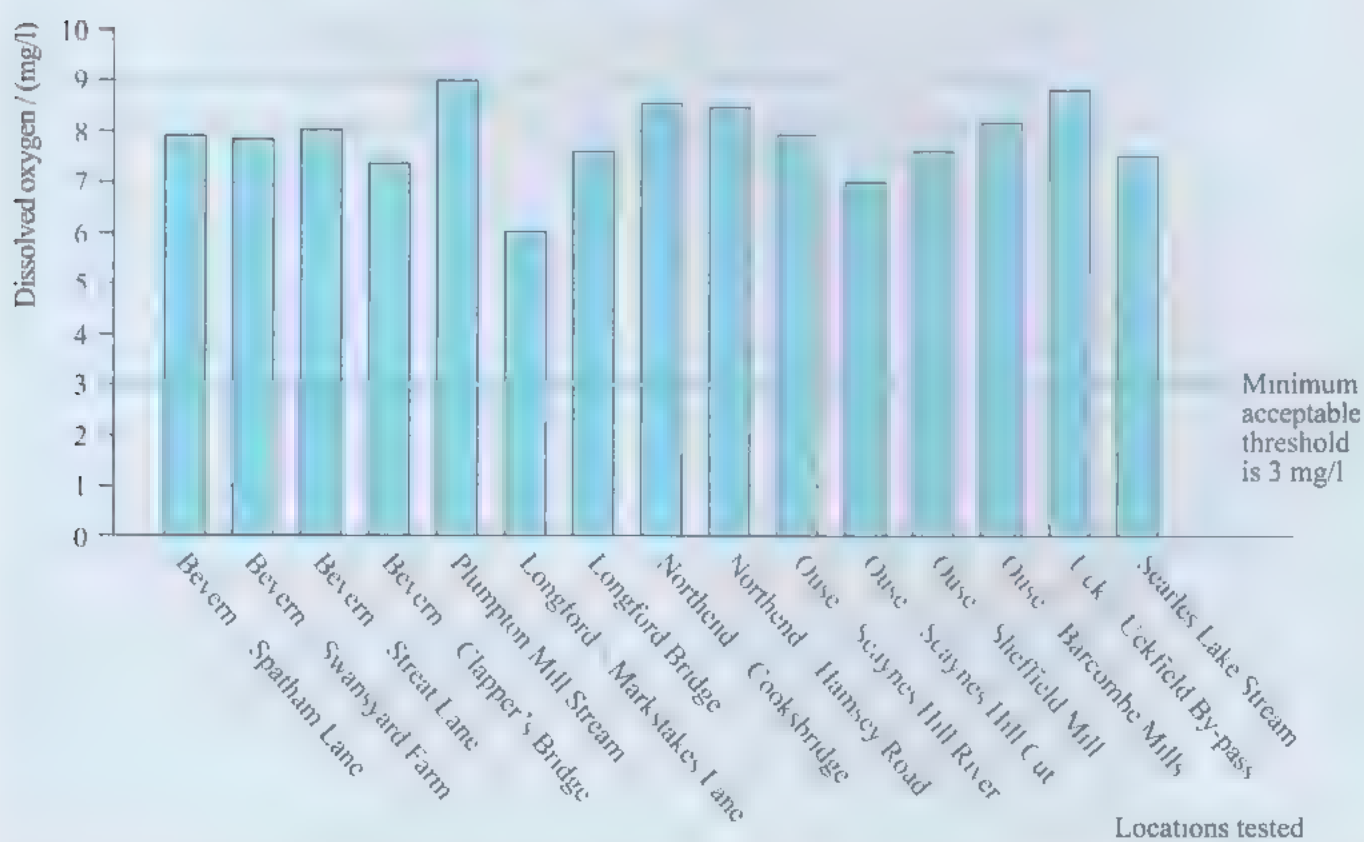
Although you can read off individual values in this way, one advantage of using a chart is that it conveys an overall picture at the same time. You can see straightaway that the location with the highest value of ammonia was on the River Bevern at Swansyard Farm and the location with the lowest value was on the River Uck at Uckfield By-pass.

The maximum acceptable level for ammonia is 0.4 mg/l and a line has been drawn on the chart to indicate this limit. The chart shows that only two of the bars are higher than this line which shows that the water at Spatham Lane and Swansyard Farm on the Bevern Stream exceeded the limit for ammonia.

Activity 190 Is there enough oxygen in the river?

The graph below shows the average amount of dissolved oxygen in the water at the different locations from November 2004 to October 2005. Oxygen is necessary for the fish to survive and the SOCS minimum acceptable threshold is 3 mg/l so Mark needs to check that all the locations have a higher value than this.

Dissolved oxygen averaged over 12 months to October 2005
Source: SOCS monthly water quality reports



(a) Did any locations fail to meet the minimum threshold level for dissolved oxygen?

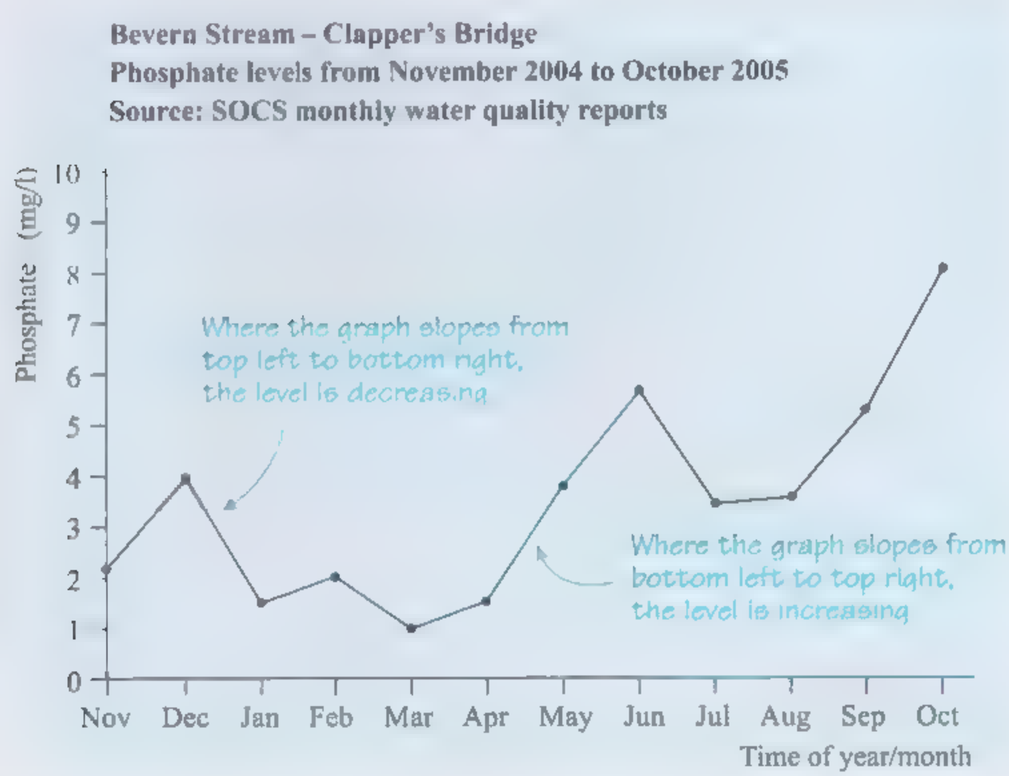
(b) What were the highest and lowest average values and where did they occur?

Comment

(a) All the bars are higher than the 3 mg/l threshold level. So the water at all the locations exceeded the minimum threshold level.

(b) The tallest bar represents the river at Plumpton Mill Stream and the level of dissolved oxygen was about 9 mg/l. The shortest bar represents the river at Longford – Markstakes Lane and the level of dissolved oxygen was about 6 mg/l.

One of the graphs that Mark and Jim discussed in the audio was a line graph which illustrated the levels of phosphate in one of the tributaries known as the Bevern Stream. The phosphate levels were recorded at the beginning of each month. The data collected were plotted as points on a graph by taking the month as the x-coordinate and the phosphate level as the y-coordinate as shown below. We say that the phosphate level has been plotted **against** time. This means that the vertical axis measures the phosphate level and the horizontal axis measures the time.



The data points were then joined by straight lines to emphasise when the readings were taken and to highlight any overall trends. Using straight lines between the points assumes that the level changed steadily between the two readings and that there were no sudden fluctuations. There is quite a lot of variation in the phosphate levels over the year, and the level of the pollutants in the water can change suddenly, for example after heavy rain. So although you can read off the coordinates of points on the graph, interpreting this information is more difficult. For example, in mid-November, the phosphate level appears to be about 3 mg/l but this assumes that the level changed steadily during November.

If all the readings were below 4 mg/l, what could you say about the mean value?

It is useful to look at the overall trend over the year. For example, in this case the phosphate level was just over 2 mg/l in November 2004 but then rose to about 4 mg/l at the beginning of December. This level then fell and appeared to stay between 1 and 2 mg/l until April. However the level rose from April reaching just below 6 mg/l at the beginning of June. By July, the level dropped to just below the 4 mg/l but climbed again during August and September, reaching the highest value for the year of about 8.5 mg/l at the beginning of October. It is clear from the graph that the phosphate level increased steadily from August onwards. This would be a cause for concern and for further investigation as Jim explained on the audio. Note that when you are interpreting an overall trend in this way, it is essential that you refer to the scales, so that you know how big the increases and decreases are. A common way of misusing graphs is either to miss off the scales or units completely, or to start the scale at some value other than zero. It is important to be clear what is being measured, which units are being used and the size of any changes before drawing any conclusions.

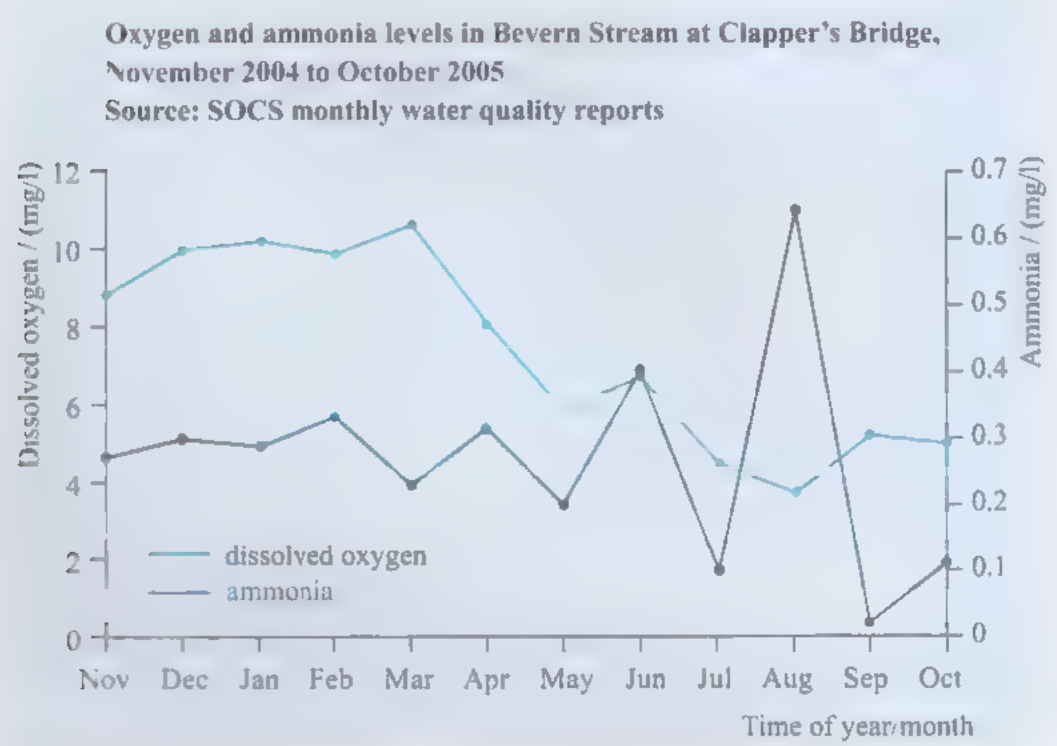
Activity 131 Interpreting the graph

From the graph of phosphate levels at Clapper's Bridge, estimate the times of the year that the phosphate concentration appeared to be above 4 mg/l. How reliable do you think these estimates are?

Comment

From the graph, the phosphate level appears to be above the 4 mg/l level from just after the beginning of May until about the end of the third week of June and then again from about the second week in August. This interpretation assumes that the phosphate level rises and falls steadily between the times the samples are taken. However, we cannot be certain that there were not other times when the level rose above 4 mg/l. For example, the level might have done so at the beginning of December.

If you look at the line graph in Activity 128, you can see that it shows the levels of three substances on a single graph, so that the levels of the different substances can be considered together. For example, if the graphs showed that the levels of ammonia and phosphate were both rising noticeably and the level of dissolved oxygen was falling, this might indicate some sort of pollution incident that would require action. Although these graphs with multiple lines look complicated, they can be analysed step by step in the same way that you analysed the phosphate graph. The graph below shows the levels of ammonia and dissolved oxygen over the year at Clapper's Bridge.



The **legend** (or **key**) at the bottom of the graph shows that the blue line represents the amount of dissolved oxygen and the black line represents the amount of ammonia. As two different quantities are shown on this graph, there are two vertical axes – you can tell which one to use from the labels on the axes. The amount of dissolved oxygen is shown on the **left** vertical axis and this axis will be used for interpreting the **blue** line that represents the amount of dissolved oxygen. The amount of ammonia is shown on the **right** vertical axis and this will be used for interpreting the **black** line that represents the amount of ammonia. For example, at the beginning of February, the level of dissolved oxygen was just under 10 mg/l and the amount of ammonia was about 0.3 mg/l.

The next activity will help you to cope with these more complicated graphs.

Activity 32 Two graphs together

From the graphs, what can you say about the water quality (for ammonia and oxygen) at Clapper's Bridge over the year?

Comment

From the beginning of November until the beginning of February, the level of dissolved oxygen was quite high (about 9 or 10 mg/l) and the level of ammonia was about 0.3 mg/l. So the water quality appears to have been reasonable over this period. During February, the ammonia level dropped and the level of dissolved oxygen rose. However from March to August the level of dissolved oxygen fell, apart from a slight increase in June. The lowest recorded reading of about 4 mg/l was obtained in August and the level then started to improve. All the readings were above the SOCS threshold of 3 mg/l. The ammonia readings fluctuated from March onwards, reaching a high level by the beginning of June (0.4 mg/l) and an exceptionally high value of about 0.65 mg/l at the beginning of August. As the dissolved oxygen level was also low during this period, the water quality was probably quite poor from mid-July to mid-August. The ammonia level had fallen by the beginning of September.

6.7 Conclusion

Graphs and charts are very useful for displaying data in a form that can be easily and quickly understood. However, different types of graphs and charts do stress different aspects of the data, so choosing an appropriate diagram to get your message across is important. For example, if you want to illustrate how a quantity changes over time, a line graph is often a good choice; if you want to show how different parts make up a whole, a pie chart is more appropriate.

Having chosen which chart or graph to use, it is essential to include sufficient detail for your reader to be able to extract the information they need easily, whether you are drawing your graph by hand or using a computer.

Drawing graphs and charts

- Include a clear title – so that your reader can see at a glance whether this chart is the one they are interested in.
- Include the source of the data – so that your reader can make further checks on how the data was collected if they wish.
- Choose the simplest graph or chart that illustrates the point you wish to emphasise.
- Label axes with both words and units.
- Mark scales clearly, choosing a scale that is easy to interpret.
- Include a legend if you have used more than one type of shading or line.

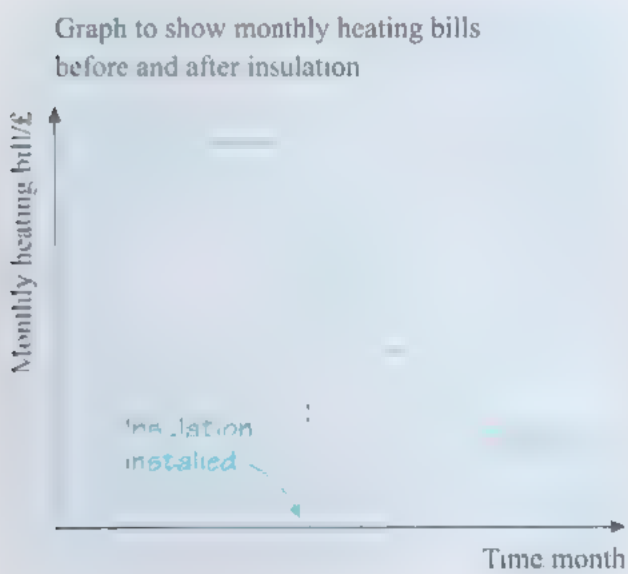
The same principles apply if you are interpreting a graph or chart.

Activity [illegible] Reading a graph with caution

The graph below shows the monthly heating bills of a house, before and after loft insulation was installed.

(a) Which of the principles in the 'Drawing graphs and charts' box have not been followed?

(b) What does the graph appear to show?



(a) No source is stated, so you have no idea how the data was collected or how reliable it may be. Was just one house or many houses used in the survey? The axes are labelled but the scales are not marked. For example suppose the scale on the vertical axis was from £50 to £51. Then the apparent drop in the bill after insulation would be negligible. However if the scale went from £0 to £50, the drop might be of more interest. No scale is marked on the horizontal axis either. In fact, the data was collected from November to September, so the horizontal scale should have indicated this.

(b) The graph appears to show a large drop in the heating bills after insulation was installed. However without the scale on the vertical axis, it is impossible to say what kind of drop this is. From the data, the drop in the monthly heating cost occurred in the May bill just as the weather was warming up for the summer. So the reduced bills could simply be due to less heating being used in the summer. Overall, no conclusions can be made from this graph about the effectiveness of the insulation.

The last activity illustrated an important point: when you are comparing two sets of data, you need to compare **like with like**. You would expect the bills for the summer to be less than those in the winter anyway. It would be more appropriate to consider the amount of energy used for heating over two periods with similar weather. Alternatively, you could compare two similar groups of houses, one group with the insulation and the other without, over the same period of time. So although graphs and charts are very useful, it is important to read them critically, checking that all the information you need to interpret them is provided.

Another important point to note when interpreting a graph is that if the graph appears to show an association between two quantities, it does not prove that one has caused the other. For example, if over the summer months the level of a pollutant in a river steadily rose and the number of dead fish collected each week from the river also rose, it is not possible to conclude that the pollutants caused the deaths of the fish without further scientific evidence. The fish may have died from some other factor such as a disease, a different pollutant or some other change in the river.

In the previous section, you looked at some graphs that are used on a regular basis to monitor the water quality of a river system. In this section you have looked at some of the difficulties you can encounter when faced with a graph or chart. Look out for other real-life graphs and charts in newspapers, magazines, on television or on the internet. Do these graphs follow the guidelines and can you understand what the graphs are illustrating?

Study checklist

You should now be able to:

- calculate the mean, median and range for a data set and understand when to use them
- interpret and construct tables
- interpret charts and graphs critically
- construct appropriate charts and graphs to display information.

7 Conclusion

This chapter looks back over the course as a whole, including what you have achieved and what you might plan to do next. There have been three main themes running through this course that we are going to consider in detail. These are:

- improving your mathematical skills, including using your calculator
- developing your approach to problems
- helping you become an effective learner.

We hope this review will enable you to see the progress you have made and also allow you to stand back and highlight the main ideas in the course. One of the key ideas in the course is that numbers can be represented in many different forms such as decimals, fractions, percentages, ratios or in scientific notation. These alternative forms may represent the same value, but they are often used for different purposes. In the same way, a relationship may be described in words, by a formula or drawn as a graph. If you have a set of data, it can often be represented by various types of graph too. Graphs, charts and tables do stress (and ignore) different aspects of the data, so depending on what you wish to explore or show, one form might be more appropriate than the others. Thinking about what is the same and what is different about these representations and the situations in which they are used, can help you to see the connections between them and deepen your overall understanding.

7.1 Your mathematical skills

During this course, you have developed several different mathematical skills, including using your calculator and learning new notation and vocabulary. In this section, we shall look at these skills in detail.

Mathematical ideas

Over the last few months, you have met many mathematical ideas and have tackled a variety of problems and investigations. These have included the following.

- Practical mathematical techniques that you can use in a lot of everyday situations, such as handling decimals, percentages, and graphs. We hope that you now feel more confident in using mathematics in your everyday life, whether you are working things out in your head, using pencil and paper or your calculator.
- Aspects of mathematics that are fascinating and fun in their own right, such as the 1089 puzzle, number puzzles, Goldbach's conjecture and the geometric puzzles using the tangram. These introduced you to the idea of proof – one of the key aspects of mathematics that you will meet again if you decide to study other mathematics courses.

- Tackling real-life problems using the modelling cycle and using mathematics to help in making decisions, whether in a DIY project or in conserving a river.
- Learning how mathematics has developed in the past, and how there are many exciting developments in mathematics today.

We hope that the course has given you a brief glimpse of the many different ways that maths is used in our lives. The next activity will help you to review your progress in these different aspects.

Activity 134 Reviewing your maths skills

Look back over the course as a whole, the study checklists and the notes you have made. What mathematical skills have you developed? Try to identify:

- the mathematical techniques where you feel you have made progress
- one puzzle or activity that you found challenging or intriguing
- an example where you have used the maths that you have learned in your everyday life.

Has your view of mathematics changed over the course?

Comment

All through the course, you have been working on activities, writing notes and thinking about your learning and we hope that you have made progress in several areas. However, you may find that some of these skills may fade over time unless you continue to use them. You can do this by looking out for maths around you, for example, practising the mental strategies when shopping, looking critically at graphs and charts in the media, tackling everyday problems using maths, and so on. Talking to other people about these ideas can help too.

Using your calculator

Being able to use a calculator effectively is a very important skill, both at home, in work or in your studies. It takes the drudgery out of calculations and allows you to concentrate on the mathematical ideas or the problem in hand. However, it is important that you understand how your calculator works so that you can be sure that any key sequence you type in does carry out the calculation you intended.

Activity 135 Calculator tips

If you were asked to pass on some tips for using a calculator to a new student on this course, what would you say?

Comment

You have probably suggested a variety of different ideas. Here are some that we feel would be important to mention.

- Learn how your calculator keys work, the order in which the calculator works out the calculations, and how it displays numbers in different forms.
- Make an estimate of the answer first before you use your calculator so you have a rough idea of the number to expect. You can do this by rounding all the numbers in the calculation to one or possibly two significant figures first and then using some of the mental strategies (or pencil and paper) to work out an estimate.
- If you need to make several calculations that depend on each other, make sure that you use full calculator accuracy throughout all the intermediate calculations and only round your final answer. Otherwise you may introduce rounding errors, making your final answer inaccurate.
- Do round your answer appropriately at the end of the calculation. Check that it agrees with the estimate and makes sense for the problem you are working on. Answers such as 'You will need to buy 3.146 666 666 chairs' are not appropriate!
- You can sometimes use your calculator to make up your own examples, to check your understanding of other topics such as fractions, percentages and scientific notation.

Although you now know how to use a lot of the keys on the calculator and have also seen how to use the instruction booklet to work out how to use some more, we have not yet explored the memory keys. These are particularly useful if you need to carry out complicated calculations that have intermediate results (as mentioned above). If you are planning on continuing with your studies in mathematics, science or technology, you might like to consider working through Chapter 7 of the Calculator booklet to learn how to do this.

Getting to grips with notation and terminology

Throughout the history of mathematics, clear, concise notation and precise terminology have played a part in helping people to think about and share ideas, as well as helping them to solve problems more easily. However, getting used to new notation and terminology is like learning a new language. It does take time and effort to master it, whether you are an Ancient Roman getting to grips with those new digits from 0 to 9, or a student suddenly exposed to some symbols that you have not met before. Practising using the new notation and terminology in your discussions with other people and in your writing will help. Activity 136 will help you to review some of the notation that you have met on this course.

Activity 130 Test the notation

During the course, you have met a lot of different types of mathematical notation, some for numbers, some for geometry and some for keys on your calculator. Some of this notation is shown below. Can you explain what it means and give an example of where it might be used? If you cannot remember, look it up in your dictionary or this book and just carry on!

Symbol	Meaning	Example	Symbol	Meaning	Example
≠			+		
≥					
≤			·		
>					
<					
≈					
$\sqrt{}$			×		
( )			÷		
⊥			[x]		
\			[Σ]		
$A\hat{C}B$			°"		
↗ ↗			[x']		
∟			[·]		
2 3			[1/x]		

Comment

As well as new notation, you have probably also met some new mathematical vocabulary, including some words that have a more general everyday meaning as well. Have a look back through the book and your notes for new mathematical terms. Can you explain what they mean? If you would find it helpful, you can compile a glossary for these words, explaining the ideas in your own words.

7.2 Your approach to problems

One thread that has run through the entire course, from Fermat's Last Theorem to Goldbach's conjecture and through many diversions in between, is that it is all right to get stuck with a mathematical problem. Grappling with some new idea or problem is often when you manage to sort out ideas, learn a lot and make new discoveries. So we hope by now that you have indeed got stuck somewhere in the course, whether on an activity on a topic you had not met before, a Brain stretcher or a margin note. That will have given you the opportunity to try out some of the different problem-solving strategies,

particularly those in Chapter 3. These strategies are like your emergency first aid kit! When a problem strikes, it should be second nature to put one of the strategies into action, whether it's writing down a list for I know I want, drawing a diagram, trying a simpler example, breaking down the problem into steps or looking back over similar examples. And if the first strategy does not work, just try another!

Activity 137 Where did you get stuck?

Thinking back over the course as a whole, were there any places where you did get stuck? If so, which strategies did you try and did they work? What did you learn from these experiences?

Comment

Everyone will have their own personal experience here. If you got completely stuck and none of the strategies seemed to work, did you contact your tutor?

The second important thread running through the course is that problems can usually be approached in many different ways, whether you are counting goats in Lesotho or estimating the area of body burns in Britain. What is important is that you choose a method that is appropriate to the situation and one that you feel confident using. In many real-life cases, a rough estimate may be all that is required, so the key question becomes 'Is this answer good enough for the purpose?' not 'Is this answer right?'. Maths in real life does tend to be messy, and assumptions often have to be made. You saw how using the mathematical modelling cycle of describing the problem; collecting information and simplifying, doing the maths and interpreting the answer does give you a framework for tackling real-life problems. You may like to try using it when faced with a mathematical problem at home or at work.

7.3 Your study skills

While studying this course, you have developed a wide range of study skills that will be useful if you decide to carry on with your studies. Although this development is an ongoing process that you have been considering throughout the course, it is now time to look back on your progress as a whole. What have you learned about your strengths and weaknesses? Do you need to do anything about these or make any changes, particularly if you have decided to continue studying?

One way of reviewing your progress is to carry out an audit on the skills you have been developing. This involves making a list of the skills and then realistically assessing your skill level based on evidence. For example: 'I know I manage my time well because I have got all my assignments in before the cut-off dates and been well prepared for the tutorials.' The next step, if things are not working as well as you would like, is to decide which

are the priorities for you to sort out and then suggest things you could do to improve the situation. For example: 'I need to leave at least a week to tackle my assignment and I must try to delegate some of the chores to other people to give myself more time.' Or if you felt that you needed to work on making connections between the different ideas, you might decide to draw a spray diagram to show how ideas are connected or discuss ideas with other people or think about how that branch of maths is used in real life.

Activity [illegible] Reviewing your study skills

Fill in the skills audit below using a scale of 1 (low) to 3 (high) to rate your skill level.

Skill	Level	How do I know?	What do I need to do next?	Main reference
Planning my time				Chapter 1
Reading techniques				Chapter 1 for skim-reading Chapter 2 for maths
Making notes				Chapter 1
Writing mathematics				Chapter 3
Extracting ideas from audio				Chapter 1 Chapters 2 and 6
Asking my own questions				Chapter 1 (and throughout the course)
Making connections				Chapter 3 Chapter 4 Chapter 5
Improving my learning				Throughout the course!

What have you learned about being a student? Do you prefer a visual practical approach using diagrams or a more systematic approach where you can read things through carefully? Do you prefer to work through step by step concentrating on the details or to have an overall view first?

Recognising your learning style may help you to study more effectively.

7.4 What next?

Now that you have nearly finished this course, we hope that you feel more confident in using mathematics, whether at home, at work or in your studies and that you are more aware of the power, the fun and the beauty of mathematics. You have also developed many different skills during the course, from learning how to use your calculator effectively to reading and writing mathematics, organising your time and learning effectively from feedback through the assignments. If you would like to continue your mathematical journey, you may like to consider studying another course, using what you have learned on this course in other areas of your life or dipping into some of the books listed below. Good luck in whatever you decide to do!

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Acknowledgements

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Text

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Figures

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Starting with Maths

Maths, as well as being a fascinating subject in its own right, also underpins many aspects of everyday life. Whether you are keeping tabs on a budget, tackling a DIY project, devising a formula to use in a spreadsheet or deciding how to present some information graphically, maths plays a part. *Starting with Maths* introduces you to a range of key ideas which will help you tackle problems mathematically whether at home, work or in your further studies. These include: understanding decimals, fractions and percentages; using formulas; measuring lengths, areas and volumes; and drawing and interpreting graphs. Sometimes you may need to do a quick calculation in your head and other times it may be easier to use your calculator – the course helps you to deal with both of these approaches. It is also about how you can develop general strategies for tackling problems and how you can overcome the feeling of 'being stuck'. Puzzles, case studies, historical snapshots and some more recent mathematical discoveries are included to illustrate both the fun and the power of maths.

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